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Output Feedback H^∞ Control for Linear Systems with Time-varying Delayed State

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Abstract This note considers the H^∞ controller design problem for linear systems with time-varying delays in states. We obtain sufficient conditions for the existence of k-th order H^∞ controllers in terms of three linear matrix inequalities(LMIs). These sufficient conditions are dependent on the maximum value of the time derivative of time-varying delay. Furthermore, we briefly explain how to construct such controllers from the positive definite solutions of their LMIs and give an example.

Keywords H^∞ control, Time-varying delay, Output Feedback, LMI

1. INTRODUCTION

Since time-delay is frequently a source of instability and encountered in various engineering systems, the stability problems of time-delay systems have received considerable attention over the decades [1]-[10]. There are many publications on solving the stabilization problem of systems with i) constant delay [2], [3], ii) time-varying delay [2], [4], iii) constant delay and parameter uncertainty [5], [6], iv) time-varying delay and parameter uncertainty [7]. The stability of the closed loop system in [3], [5], and [6] is independent of time delay, but one in [2], [4], and [7] is dependent on only the maximum value of the time derivative of time-varying delay.

The H^∞ controller design for delay systems is also interesting problem. In the frequency domain, Lee *et al.* [9] and Choi *et al.* [10] considered memoryless H^∞ state feedback controllers for state delayed systems and both state and control delayed systems, respectively. But when all state variables are not available for the feedback, these methods cannot be applied. And they did not deal with the time-varying delay case.

In this note, we consider the H^∞ output feedback controller design problem for linear systems with time-varying delays in states. Our aim is an extension of [9] to the output feedback and the time-varying delays case. The approach adopted here is based on Lyapunov functionals due to Krasovskii [1], [11] and the ideas proposed by Gahinet *et al.*[12] and Iwasaki *et al.*[13]. We obtain sufficient conditions for the existence of an H^∞ output feedback controller of any order in terms of three linear matrix inequalities(LMIs). Finally, we give a small example to illustrate the validity of the proposed design procedure.

2. PROBLEM FORMULATION AND SUFFICIENT CONDITION

Consider a state delayed system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h(t)) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \\ x(t) &= 0, \quad t \le 0 \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $w(t) \in R^l$ is the square-integrable disturbance input, $u(t) \in R^m$ is the control, $z(t) \in R^p$ is the controlled output, $y(t) \in R^q$ is the measurement output, $h(t)$ is the time-varying delay with the following assumption:

$$0 \le h(t) < \infty, \quad \dot{h}(t) \le m < 1, \tag{2}$$

and A, A_h, B_1, B_2, C_1, C_2, D_{11}, D_{12}, and D_{21} are constant matrices with appropriate dimensions. And we assume that $(A,\ B_2,\ C_2)$ is stabilizable and detectable. As an H^∞ controller of the state delayed system (1), we consider a dynamic output feedback law

$$\begin{aligned} \dot{\xi}(t) &= A_K \xi(t) + B_K y(t) \\ u(t) &= C_K \xi(t) + D_K y(t) \end{aligned} \tag{3}$$

where $\xi(t) \in R^k$ is the state of the controller and all matrices are constant with proper dimensions. When we apply the control (3) to the state delayed system (1), the closed loop system from w to z is given by

$$\begin{aligned} \dot{\eta}(t) &= A_{cl}\eta(t) + A_{clh}\eta(t-h(t)) + B_{cl}w(t) \\ z(t) &= C_{cl}\eta(t) + D_{cl}w(t) \\ \eta(t) &= 0, \quad t \le 0 \end{aligned} \tag{4}$$

where

$$\begin{aligned} \eta(t) &= \begin{bmatrix} x^T(t) & \xi^T(t) \end{bmatrix}^T, \\ A_{cl} &= \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}, \\ A_{clh} &= \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix}, \\ B_{cl} &= \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix}, \\ C_{cl} &= [C_1 + D_{12} D_K C_2 \quad D_{12} C_K], \\ D_{cl} &= D_{11} + D_{12} D_K D_{21}. \end{aligned} \tag{5}$$

Here, we gather all controller parameters into the single variable

$$K := \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \tag{6}$$

and introduce the shorthands:

$$\begin{aligned} A_0 &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_h \\ 0 \end{bmatrix}, \quad E = [I \ 0], \\ B_0 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{00} = \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix}, \\ C_0 &= [C_1 \ \ 0], \quad C_{00} = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}, \\ D_1 &= [D_{12} \ \ 0], \quad D_2 = \begin{bmatrix} D_{21}^T & 0 \end{bmatrix}^T, \end{aligned} \tag{7}$$

then the closed loop matrices A_{cl}, A_{clh}, B_{cl}, C_{cl}, and D_{cl} can be written as

$$\begin{aligned} A_{cl} &= A_0 + B_{00}KC_{00}, \\ A_{clh} &= A_1E, \\ B_{cl} &= B_0 + B_{00}KD_2, \\ C_{cl} &= C_0 + D_1KC_{00}, \\ D_{cl} &= D_{11} + D_1KD_2. \end{aligned} \tag{8}$$

Note that (7) involves only plant data and that A_{cl}, B_{cl}, C_{cl}, and D_{cl} are affine form of the controller data K. We consider the design of a stabilizing controller data K which yields the delayed closed-loop system with H^∞ norm bounded above by a specified number $\gamma > 0$.

Lemma 1 Consider a state delayed system

$$\dot{\eta}(t) = A_{cl}\eta(t) + A_1E\eta(t - h(t)) \tag{9}$$

with the assumption (2) and define $\bar{Q} = (1 - m)Q$. If there exist positive definite matrices P and Q such that

$$A_{cl}^TP + PA_{cl} + PA_1\bar{Q}^{-1}A_1^TP + E^TQE < 0, \tag{10}$$

then the system (9) is asymptotically stable.

Proof: Define a Lyapunov functional as

$$V(\eta(t), t) := \eta^T(t)P\eta(t) + \int_{t-h(t)}^{t} \eta^T(\tau)E^TQE\eta(\tau)d\tau, \tag{11}$$

then it follows from (10) easily that

$$\begin{aligned} \dot{V}(\eta(t), t) &< -\eta^T(t)PA_1\bar{Q}^{-1}A_1^TP\eta(t) \\ &\quad +\eta^T(t - h(t))E^TA_1^TP\eta(t) \\ &\quad +\eta^T(t)PA_1E\eta(t - h(t)) \\ &\quad -\eta^T(t - h(t))E^T\bar{Q}E\eta(t - h(t)) \\ &= -[A_1^TP\eta(t) - \bar{Q}E\eta(t - h(t))]^T\bar{Q}^{-1} \\ &\quad \times[A_1^TP\eta(t) - \bar{Q}E\eta(t - h(t))] \\ &\leq 0. \end{aligned}$$

So the system (9) is asymptotically stable. Q.E.D.

Lemma 2 Consider (4) with the assumption (2) and suppose that $\sigma_{max}(D_{cl}) < \gamma$. If there exist positive definite matrices P and Q such that

$$\begin{aligned} &A_{cl}^TP + PA_{cl} + PA_1\bar{Q}^{-1}A_1^TP + E^TQE \\ &+\gamma^{-2}C_{cl}^TC_{cl} + (\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP)^T \\ &\times(I - \gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP) < 0, \end{aligned} \tag{12}$$

then (4) is asymptotically stable and $\|z(t)\|_2 < \gamma\|w(t)\|_2$.

Proof: The positive definite matrices P and Q which satisfy the inequality (12) also satisfy the inequality (10). In order to establish the upper bound $\gamma\|w(t)\|_2$ for $\|z(t)\|_2$, we introduce

$$J_1 = \int_0^\infty [\gamma^{-2}z^T(t)z(t) - w^T(t)w(t)]dt, \tag{13}$$

$$J_2 = \int_0^\infty [\gamma^{-2}z^T(t)z(t) - w^T(t)w(t) + \dot{V}(\eta(t), t)]dt. \tag{14}$$

Since $J_1 \leq J_2$, the proof is completed if $J_2 < 0$. It follows from (12) easily that

$$\begin{aligned} J_2 &< \int_0^\infty \{-\eta^T(t)(\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP)^T \\ &\quad \times(I - \gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP)\eta(t) \\ &\quad +w^T(t)(\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP)\eta(t) \\ &\quad +\eta^T(t)(\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP)w(t) \\ &\quad -w^T(t)(I - \gamma^{-2}D_{cl}^TD_{cl})w(t) \\ &\quad -\eta^T(t)PA_1\bar{Q}^{-1}A_1^TP\eta(t) \\ &\quad +\eta(t - h(t))E^TA_1^TP\eta(t) \\ &\quad +\eta^T(t)PA_1E\eta(t - h(t)) \\ &\quad -\eta^T(t - h(t))E^T\bar{Q}E\eta(t - h(t))\}dt \\ &= \int_0^\infty \{\Gamma_1^T(I - \gamma D_{cl}^TD_{cl})^{-1}\Gamma_1 - \Gamma_2^T\bar{Q}^{-1}\Gamma_2\}dt \\ &\leq 0 \end{aligned}$$

where

$$\begin{aligned} \Gamma_1 &= (\gamma^{-2}D_{cl}^TC_{cl} + B_{cl}^TP)\eta(t) - (I - \gamma^{-2}D_{cl}^TD_{cl})w(t) \\ \Gamma_2 &= A_1^TP\eta(t) - \bar{Q}E\eta(t - h(t)) \end{aligned}$$

That is, $\|z(t)\|_2 < \gamma\|w(t)\|_2$. Q.E.D.

3. EXISTENCE CONDITION OF H^∞ CONTROLLERS

In this section, we present sufficient conditions for the existence of γ-suboptimal H^∞ controllers of any order and parameterize γ-suboptimal H^∞ controllers in the state space using the positive definite solutions of LMIs.

Using the LMI representation, (12) can be changed to the LMI form as

$$\begin{bmatrix} S_{cl} & PB_{cl} & C_{cl}^T & PA_1 \\ B_{cl}^TP & -\gamma I & D_{cl}^T & 0 \\ C_{cl} & D_{cl} & -\gamma I & 0 \\ A_1^TP & 0 & 0 & -\bar{Q} \end{bmatrix} < 0 \tag{15}$$

where $S_{cl} = A_{cl}^TP + PA_{cl} + E^TQE$. Equivalently, this condition with the notation of (8) can be represented as

$$\Phi + \Sigma\Pi K\Theta^T + \Theta K^T\Pi^T\Sigma^T < 0 \tag{16}$$

where

$$\begin{aligned} \Sigma &= Diag(P, \ I, \ I, \ I), \\ \Pi &= [B_{00}^T \ \ 0 \ \ D_1^T \ \ 0]^T, \\ \Theta &= [C_{00} \ \ D_2 \ \ 0 \ \ 0]^T, \end{aligned} \tag{17}$$

and

$$\Phi = \begin{bmatrix} S_0 & PB_0 & C_0^T & PA_1 \\ B_0^T P & -\gamma I & D_{11}^T & 0 \\ C_0 & D_{11} & -\gamma I & 0 \\ A_1^T P & 0 & 0 & -\bar{Q} \end{bmatrix} \tag{18}$$

$$S_0 = A_0^T P + PA_0 + E^T Q E$$

(16) is solvable for some K if and only if

$$\Pi_\perp^T \Sigma^{-1} \Phi \Sigma^{-1} \Pi_\perp < 0, \tag{19}$$

$$\Theta_\perp^T \Phi \Theta_\perp < 0, \tag{20}$$

where $\Pi_\perp$ and $\Theta_\perp$ are orthogonal complements of Π and Θ, respectively [11]-[13]. Using the conditions (19) and (20), we can eliminate the controller data K to obtain conditions including only P. To simplify the conditions (19) and (20), we partition P and P^{-1} as

$$P = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}, \tag{21}$$

where $X,\ Y \in R^{n\times n}$, $M,\ N \in R^{n\times k}$, and $*$ means irrelevant. And we can choose $[W_1^T \ \ W_2^T]^T$ and $[W_3^T \ \ W_4^T]^T$ which are orthogonal complements of $[B_2^T \ \ D_{12}^T]^T$ and $[C_2 \ \ D_{21}]^T$, repectively, then

$$\Pi_\perp = \begin{bmatrix} W_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ W_2 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \Theta_\perp = \begin{bmatrix} W_3 & 0 & 0 \\ 0 & 0 & 0 \\ W_4 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{22}$$

The inequalities (19) and (20) are simplified to

$$\tilde{\Pi}^T \tilde{X} \tilde{\Pi} < 0, \tag{23}$$

$$\tilde{\Theta}^T \tilde{Y} \tilde{\Theta} < 0 \tag{24}$$

where

$$\tilde{\Pi} = \begin{bmatrix} W_1 & 0 & 0 \\ W_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{\Theta} = \begin{bmatrix} W_3 & 0 & 0 \\ W_4 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$\tilde{X} = \begin{bmatrix} XA^T + AX + XQX & XC_1^T & B_1 & A_h \\ C_1 X & -\gamma I & D_{11} & 0 \\ B_1^T & D_{11}^T & -\gamma I & 0 \\ A_h^T & 0 & 0 & -\bar{Q} \end{bmatrix}$$

$$\tilde{Y} = \begin{bmatrix} A^T Y + YA + Q & YB_1 & C_1^T & YA_h \\ B_1^T Y & -\gamma I & D_{11}^T & 0 \\ C_1 & D_{11} & -\gamma I & 0 \\ A_h^T Y & 0 & 0 & -\bar{Q} \end{bmatrix}.$$

Since $Q > 0$, (23) is equivalent to

$$\bar{\Pi}^T \bar{X} \bar{\Pi} < 0 \tag{25}$$

where

$$\bar{\Pi} = \begin{bmatrix} W_1 & 0 & 0 & 0 \\ W_2 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

$$\bar{X} = \begin{bmatrix} XA^T + AX & XC_1^T & B_1 & A_h & X \\ C_1 X & -\gamma I & D_{11} & 0 & 0 \\ B_1^T & D_{11}^T & -\gamma I & 0 & 0 \\ A_h^T & 0 & 0 & -\bar{Q} & 0 \\ X & 0 & 0 & 0 & -Q^{-1} \end{bmatrix}.$$

Theorem 1 Consider the system (1) with the assumption (2) and let $[W_1^T \ \ W_2^T]^T$ and $[W_3^T \ \ W_4^T]^T$ are orthogonal complements of $[B_2^T \ \ D_{12}^T]^T$ and $[C_2 \ \ D_{21}]^T$, repectively. If there exist positive definite matrices X and Y satisfying the LMIs (25) and (24), respectively, and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{26}$$

for some $Q > 0$, then the γ-suboptimal H^∞ control problem is solvable.

Proof: $P > 0$ if and only if the inequality $X - Y^{-1} \geq 0$ holds. This inequality is equivalent to (26). The rest of the proof is mentioned before. Q.E.D.

Note that theorem 1 does not present the computation of the controller itself, but existence conditions of H^∞ controllers. To compute H^∞ controllers, firstly compute some solutions $(X,\ Y)$ satisfying the LMIs (24)-(26), secondly compute two full-column-rank matrices $M,\ N \in R^{n\times k}$ such that

$$MN^T = I - XY. \tag{27}$$

Then the unique solution P is obtained from the linear equation:

$$\begin{bmatrix} Y & I \\ N^T & 0 \end{bmatrix} = P \begin{bmatrix} I & X \\ 0 & M^T \end{bmatrix}. \tag{28}$$

Note that (28) is always solvable when $Y > 0$ and M has full column rank[14]. Given P, since (16) is an LMI in K, γ-suboptimal H^∞ controllers can be computed as any solution K of (16). Because the order of the controller depends on the dimension of P, we can establish the following corollary.

Corollary 1 Suppose that the γ-suboptimal H^∞ control problem for the system (1) is solvable. If

$$Rank(I - XY) = k < n \tag{29}$$

for some $X > 0$, $Y > 0$ satisfying (24)-(26), then there exist γ-suboptimal H^∞ controllers of order k. Q.E.D.

Remark 1 In lemma 1, 2, and theorem 1, the derived conditions are dependent on the maximum value of the time derivative of time-varying delay. In the constant delay case, lemma 1, 2, and theorem 1 are independent of time delay because the time derivative of time delay is zero.

Remark 2 Note that lemma 1, 2, and theorem 1 can be easily extended to the multiple time-varying delays case, choosing a similar Lyapunov functional to the one proposed in [1] and [11].

Example Consider a system of (1) with

$$A=\begin{bmatrix}-2 & 1\\ -1 & 1\end{bmatrix},\quad A_h=\begin{bmatrix}0.2 & 0.1\\ 0.3 & 0.1\end{bmatrix},$$

$$B_1=\begin{bmatrix}1\\ 0\end{bmatrix},\quad B_2=\begin{bmatrix}0\\ 1\end{bmatrix},$$

$$C_1=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix},\quad C_2=[1\ 3],$$

$$D_{11}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\quad D_{12}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},\quad D_{21}=1,$$

$$h(t)=5+0.5sin(t).$$

Let $\gamma=2$, $m=0.5$, and $Q=I_2$, then one pair of the positive definite solutions satisfying (24)-(26) is

$$(X,Y)=\left(\begin{bmatrix}1.8121 & 0.1500\\ 0.1500 & 0.5537\end{bmatrix},\begin{bmatrix}3.8725 & 0.8445\\ 0.8445 & 2.1685\end{bmatrix}\right),$$

and one pair of the solutions satisfying (27) is

$$(M,N)=\left(\begin{bmatrix}-0.9858 & -0.1682\\ -0.1682 & 0.9858\end{bmatrix},\begin{bmatrix}6.2328 & 0\\ 1.8841 & -0.0107\end{bmatrix}\right).$$

The positive definite solution of (28) is

$$P=\begin{bmatrix}3.8725 & 0.8445 & 6.2328 & 0\\ 0.8445 & 2.1685 & 1.8841 & -0.0107\\ 6.2328 & 1.8841 & 11.7449 & -0.0026\\ 0 & -0.0107 & -0.0026 & 0.0056\end{bmatrix}$$

and one of the H^∞ controllers satisfying (16) is

$$K=\left[\begin{array}{c|c}D_K & C_K\\ \hline B_K & A_K\end{array}\right]=\left[\begin{array}{c|cc}-1.3051 & -4.2301 & 0.0858\\ \hline -0.3660 & -2.7411 & 0.0027\\ 1.2079 & 1.5247 & -8.4108\end{array}\right].$$

4. CONCLUSION

In this note, we have developed an H^∞ output feedback controller design method for linear systems with time-varying delays in states. We have obtained sufficient conditions for the existence of k-th order H^∞ controllers in terms of three LMIs. And we have briefly explain how to construct such controllers from the positive definite solutions of their LMIs. The H^∞ output feedback controller guarantees not only the asymptotic stability of the closed loop system but also the H^∞ norm bound.

REFERENCES

[1] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications*, North-Holland Systems and Control Series, vol. 9, Amsterdam, 1987.

[2] A. Feliachi and A. Thowsen, "Memoryless stabilization of linear delay-differential systems," *IEEE Trans. Automat. Contr.*, vol. 26, no. 2, pp. 586-587, 1981.

[3] T. Mori, E. Noldus, and M. Kuwahara "A way to stabilize linear systems with delayed state," *Automatica*, vol. 19, no. 5, pp. 571-573, 1983.

[4] M. Ikeda and T. Ashida, "Stabilization of linear systems with time-varying delay," *IEEE Trans. Automat. Contr.*, vol. 24, no. 2, pp. 369-370, 1979.

[5] M. S. Mahmoud and N. F. Al-Muthairi, "Design of robust controllers for time-delay systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 5, pp. 995-999, 1994.

[6] J. C. Shen, B. S. Chen, and F. C. Kung, "Memoryless stabilization of uncertain dynamic delay systems: Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 36, no. 5, pp. 638-640, 1991.

[7] H. H. Choi and M. J. Chung, "Memoryless stabilization of uncertain dynamic systems with time-varying delayed states and controls," *Automatica*, vol. 31, no. 9, pp. 1349-1351, 1995.

[8] K.-K. Shyu and J.-J. Yan, "Robust stability of uncertain time-delay systems and its stabilization by variable structure control," *Int. J. Control*, vol. 57, no. 1, pp. 237-246, 1993.

[9] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H^∞ controllers for state delayed systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 159-162, 1994.

[10] H. H. Choi and M. J. Chung, "Memoryless H_∞ controller design for linear systems with delayed state and control," *Automatica*, vol. 31, no. 6, pp. 917-919, 1995.

[11] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

[12] P. Gahinet and P. Apkarian, "An LMI-based parameterization of all H_∞ controllers with applications," in *Proc. IEEE Conf. Dec. Contr.*, pp. 656-661, 1993.

[13] T. Iwasaki and R. E. Skelton, "All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas," *Automatica*, vol. 30, no. 8, pp. 1307-1317, 1994.

[14] A. Packard, K. Zhou, P. Pandey, and G. Becker, "A collection of robust control problems leading to LMI's," in *Proc. IEEE Conf. Dec. Contr.*, pp. 1245-1250, 1991.

[15] I. R. Petersen, "Disturbance attenuation and H^∞ optimization: a design method based on the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, vol. 32, no. 5, pp. 427-429, 1987.