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OPERATIONS OF FUZZY BAGS

°Kyung-Soo Kim* and Sadaaki Miyamoto**

*Doctoral Program in Engineering, University of Tsukuba, Ibaraki 305, JAPAN
Tel: +81-298-53-6553; Fax: +81-298-53-6471; Email: kim@odin.esys.tsukuba.ac.jp

**Institute of Information Sciences and Electronics, University of Tsukuba, Ibaraki 305, JAPAN
Tel: +81-298-53-5346; Fax: +81-298-53-6471; Email: miyamoto@esys.tsukuba.ac.jp

Abstract A bag is a set-like entity which can contain repeated elements. Fuzzy bags have been studied by Yager, who defined their basic relations and operations. However, his definitions of the basic relations and operations are inconsistent with the corresponding relations and operations for ordinary fuzzy sets. The present paper presents new basic relations and operations of fuzzy bags using a *grade sequence* for each element of the universal set. Moreover the α-cut, t-norms, the extension principle, and the composition of fuzzy bag relations are described.

Keywords fuzzy bag, grade sequence, union and intersection, α-cut, bag relation.

1. INTRODUCTION

To deal with information systems nowadays, sometimes new mathematical frameworks are required. For example, relational database systems are now standard, but some capabilities of its query language cannot appropriately be described by the traditional mathematical tool. Namely, the SQL [8] query language allows multiple copies of the same tuple in a derived table, whereas the relational database is based on the classical set theory which do not permit the existence of such multiple copies.

Bags [5] which are also called multisets [2] is a set-like entity which can contain repeated elements, and therefore suitable as a framework to handle the above system. Since fuzzy relational database systems have widely been studied (cf. e.g., [9, 12]), the necessity of a fuzzy version of bags seems to be obvious; indeed, Yager [10] has introduced fuzzy bags and discussed its basic relations and operations.

We notice, however, that these relations and operations are insufficient in order to be useful in applications. We therefore introduce new basic relations and operations which clarify relationships of compatibility among fuzzy bags, ordinary fuzzy sets, and crisp bags. The key idea is to use a standard form that is called *a grade sequence form* here.

An important operation in fuzzy sets is the α-cut and the commutativity of the α-cut with other operations. For the commutativity for ordinary fuzzy sets, see [1, 6]. Here the α-cut of fuzzy bags is presented. Moreover other operations such as the t-norms [3], the extension principle, and the composition of fuzzy bag relations are described. Proofs of the theorems are mostly omitted to save the space. Some of them are found in Miyamoto [7] and others are obvious when the basic idea is understood.

2. FUZZY BAGS

2.1 Bags and Fuzzy Bags

It should be remarked that throughout this paper we use the term of *bags* instead of *multisets*. However, we assume that the both terms have the same meaning.

Let $X = \{x_1, ..., x_n\}$ be a finite universal set and assume that bags are considered in this set. Let $\mathbf{N} = \{0, 1, 2, ...\}$ be the set of natural numbers and $I = [0, 1]$ be the unit interval. A (crisp) bag, say M, is characterized by a function $C_M: X \to \mathbf{N}$. When $C_M(x_i) = n_i$, it means that there are n_i copies of x_i in N. For example, let $X = \{a, b, c\}$ and $M = \{a, a, a, c\}$, then $C_M(a) = 3$, $C_M(b) = 0$, and $C_M(c) = 1$. We can express $M = \{3/a, 0/b, 1/c\}$. Generally, we write $M = \{C_M(x)/x : x \in X\}$.

The inclusion and equality between two bags L and M are defined to be

$$L \subseteq M \iff C_L(x) \le C_M(x),\ \forall x \in X, \quad (1)$$
$$L = M \iff C_L(x) = C_M(x),\ \forall x \in X. \quad (2)$$

The union and intersection are given by

$$C_{L\cup M}(x) = C_L(x) \vee C_M(x),\ \forall x \in X, \quad (3)$$
$$C_{L\cap M}(x) = C_L(x) \wedge C_M(x),\ \forall x \in X. \quad (4)$$

Notice that there are resemblances between the bag operations and fuzzy set operations.

Fuzzy bags have been introduced by Yager [10], who defines a fuzzy bag of X by a crisp bag of $X \times I$. When $A = \{C_A(x, \mu)/(x, \mu) : x \in X, \mu \in I\}$ is a fuzzy bag, it means that the number of x with the membership degree μ is $C_A(x, \mu)$. For example, let $A = \{1/(a, 0.7), 2/(a, 0.3), 1/(b, 0.5)\}$. This means that there are three a's with the degrees 0.7, 0.3, 0.3, and one b with the membership 0.5. Hence we can write $A = \{(a, 0.7), (a, 0.3), (a, 0.3), (b, 0.5)\}$.

The definitions of the inclusion, equality, union, and intersection of fuzzy bags introduced by Yager are those of crisp bags of $X \times I$.

There are drawbacks in these definitions. First, fuzzy bag operations are incompatible with those of ordinary fuzzy sets. To see this, let us introduce an imbedding operator $\Gamma(\cdot)$ that maps an ordinary fuzzy set into the fuzzy bag of the same content. Assume $A = \sum_i \mu_i/x_i$. Then, define

$$\Gamma(A) = \{1/(x_i, \mu_i)\}_i \tag{5}$$

where $\mu_i \neq 0$.

Consider an example in which $E = 0.1/a + 0.6/b$ and $F = 0.5/b$. Then, $\Gamma(E) = \{1/(a, 0.1), 1/(b, 0.6)\}$ and $\Gamma(F) = \{1/(b, 0.5)\}$. By the operation by Yager,

$$\Gamma(E) \cup \Gamma(F) = \{1/(a, 0.1), 1/(b, 0.6), 1/(b, 0.5)\},$$

while $E \cup F = E$.

In general, the compatibility between the fuzzy bag relations and operations and those of ordinary fuzzy sets are expressed as

$$\Gamma(A) \subseteq \Gamma(B) \iff A \subseteq B, \tag{6}$$
$$\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B), \tag{7}$$
$$\Gamma(A) \cap \Gamma(B) = \Gamma(A \cap B), \tag{8}$$

where A and B are ordinary fuzzy sets. The above example shows that these relations do not hold.

Second, it is difficult to obtain the operation of α-cut from the definitions by Yager. Notice that we wish to define the α-cut so that the resulting crisp bags have another set of compatibility relations. Namely, let A and B be two fuzzy bags and suppose that we have appropriately defined the α-cut A_α. Then we wish to obtain the following relations (cf. [1, 6]).

$$A \subseteq B \iff A_\alpha \subseteq B_\alpha,\ \forall \alpha \in (0, 1], \tag{9}$$
$$(A \cup B)_\alpha = A_\alpha \cup B_\alpha, \tag{10}$$
$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha. \tag{11}$$

With these motivations in mind, we introduce a standard form. First, we collect the membership degrees for each object $x \in X$:

$$A = \{n_i/(x_i, \mu_i)\}_{i=1,..,n} \tag{12}$$
$$= \{\{\mu_i', ..., \nu_i'\}/x_i\}_{i=1,..,n} \tag{13}$$

For the above example,

$$A = \{\{0.3, 0.7, 0.3\}/a, \{0.5\}/b\}.$$

Moreover, we arrange these degrees $\{\mu_i', ..., \nu_i'\}$ into the decreasing order: $\mu_i' \geq ... \geq \nu_i'$. The resulting sequence is called the grade sequence for x_i. We write, instead of $\mu_i', ..., \nu_i'$; $\mu_A^1(x), ..., \mu_A^p(x)$ for $x \in X$ ($\mu_A^1(x) \geq ... \geq \mu_A^p(x)$) and assume that p is independent of x by appending appropriate numbers of zero degrees for simplicity. Thus,

$$A = \{(\mu_A^1(x), ..., \mu_A^p(x))/x \ :\ x \in X\}. \tag{14}$$

For example,

$$A = \{\{0.3, 0.7, 0.3\}/a, \{0.5\}/b\}$$
$$= \{(0.7, 0.3, 0.3)/a, (0.5, 0, 0)/b, (0, 0, 0)/c\}$$

in this form.

2.2 Basic Operations of Fuzzy Bags

Now, we can define the basic relations and operations as follows, assuming A and B are two fuzzy bags of X.

(I) [inclusion]

$$A \subseteq B \iff \mu_A^j(x) \leq \mu_B^j(x),\ j = 1, 2, ..., p,\ \forall x \in X.$$

(II) [equality]

$$A = B \iff \mu_A^j(x) = \mu_B^j(x),\ j = 1, 2, ..., p,\ \forall x \in X.$$

(III) [union]

$$\mu_{A \cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1, 2, ..., p,\ \forall x \in X.$$

(IV) [intersection]

$$\mu_{A \cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1, 2, ..., p,\ \forall x \in X.$$

(V) [α-cut]
For arbitrarily given $\alpha \in (0, 1]$, the number of copies of x in A_α, the α - cut of A, is defined by

$$C_{A_\alpha}(x) = 0 \iff \mu_A^1(x) < \alpha$$
$$C_{A_\alpha}(x) = k \iff \mu_A^k(x) \geq \alpha \text{ and } \mu_A^k(x) < \alpha,\ (k < p)$$
$$C_{A_\alpha}(x) = p \iff \mu_A^p(x) \geq \alpha$$

We have the following theorems, the proofs of which are mostly omitted.

Theorem 1. For ordinary fuzzy sets A and B and the above introduced relations and operations (I–IV), the relations (6), (7), and (8) hold.

Theorem 2. For fuzzy bags A and B and the above introduced relations and operations (I–IV), the relations (9), (10), and (11) hold.

Theorem 3. Fuzzy bags satisfy

(a) [the commutative law]

$$A \cup B = B \cup A, \quad A \cap B = B \cap A.$$

(b) [the associative law]

$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

(c) [the distributive law]

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Thus, the class of all fuzzy bags of X forms a distributive lattice [4].

We show the proof of the distributive law in order to see how the α-cut works. Let us note that the last law holds if and only if

$$[A \cap (B \cup C)]_\alpha = [(A \cap B) \cup (A \cap C)]_\alpha \qquad (15)$$

is valid for all $\alpha \in (0,1]$, remembering (9). To show (15), we see that the followings hold.

$$\begin{aligned} [A \cap (B \cup C)]_\alpha &= A_\alpha \cap (B \cup C)_\alpha \\ &= A_\alpha \cap (B_\alpha \cup C_\alpha) \\ &= (A_\alpha \cap B_\alpha) \cup (A_\alpha \cap C_\alpha) \\ &= (A \cap B)_\alpha \cup (A \cap C)_\alpha \\ &= [(A \cap B) \cup (A \cap C)]_\alpha, \end{aligned}$$

Notice the relations (10), (11), and that the distributive law is valid for crisp bags [5].

2.3 Other Operations

The t-norms and the extension principle can be defined using the grade sequence form.

Let a t-norm and an s-norm (also called t-conorm) be denoted by $t(a,b)$ and $s(a,b)$, respectively. When we consider ordinary fuzzy sets, say E and F, the generalizations of the union and intersection are given by s-norms and t-norms:

$$\begin{aligned} \mu_{E \cup_s F}(x) &= s(\mu_E(x), \mu_F(x)), \qquad (16) \\ \mu_{E \cap_t F}(x) &= t(\mu_E(x), \mu_F(x)). \qquad (17) \end{aligned}$$

It is straightforward to extend these operations to the case of fuzzy bags, since we have obtained the grade sequence form. Namely,

(VI) [generalized union by s-norm]

$$\mu^j_{A \cup_s B}(x) = s(\mu^j_A(x), \mu^j_B(x)),\ j = 1, 2, ..., p,\ \forall x \in X.$$

(VII) [generalized intersection by t-norm]

$$\mu^j_{A \cap_t B}(x) = t(\mu^j_A(x), \mu^j_B(x)),\ j = 1, 2, ..., p,\ \forall x \in X.$$

We have

Theorem 4. These generalized operations are well-defined, that is,

$$\mu^1_{A \cup_s B}(x) \geq \mu^2_{A \cup_s B}(x) \geq ... \geq \mu^p_{A \cup_s B}(x) \qquad (18)$$

$$\mu^1_{A \cup_t B}(x) \geq \mu^2_{A \cup_t B}(x) \geq ... \geq \mu^p_{A \cup_t B}(x) \qquad (19)$$

Remark. The standard union by max and intersection by min themselves are an s-norm and a t-norm. The above theorem therefore implies that these standard operations are well-defined.

Let X and Y be two universal sets and $f: X \to Y$ be a function. Let us first discuss the inverse image and image of crisp bags. Let L (resp. M) be a crisp bag of X (resp. Y). We define

$$C_{f^{-1}(M)}(x) = C_M(f(x)),\ \forall x \in X, \qquad (20)$$

$$C_{f(L)}(y) = \bigvee_{x \in f^{-1}(y)} C_L(x),\ \forall y \in Y. \qquad (21)$$

($C_{f(L)}(y) = 0$ for $y \notin f(X)$.)

Now, assume that A (resp. B) is a fuzzy bag of X (resp. Y). We define as follows.

(VIII) [extension principle]

$$\mu^j_{f^{-1}(B)}(x) = \mu^j_B(f(x)),\ j = 1, 2, ..., p,\ \forall x \in X,$$

$$\mu^j_{f(A)}(y) = \bigvee_{x \in f^{-1}(y)} \mu^j_A(x),\ j = 1, ..., p,\ \forall y \in Y.$$

When $y \notin f(X)$, we put $\mu^j_{f(A)}(y) = 0$, $j = 1, 2, ..., p$.

Theorem 5. $f^{-1}(B)$ and $f(A)$ given in (VIII) are well-defined. Namely,

$$\mu^1_{f^{-1}(B)}(x) \geq \mu^2_{f^{-1}(B)}(x) \geq ... \geq \mu^p_{f^{-1}(B)}(x) \qquad (22)$$

$$\mu^1_{f(A)}(y) \geq \mu^2_{f(A)}(y) \geq ... \geq \mu^p_{f(A)}(y) \qquad (23)$$

for all $x \in X$ and $y \in Y$.

Theorem 6. The following relations (the commutativity with the α-cut) hold:

$$\begin{aligned} \{f^{-1}(B)\}_\alpha &= f^{-1}(B_\alpha),\ \forall \alpha \in (0,1], \qquad (24) \\ \{f(A)\}_\alpha &= f(A_\alpha),\ \forall \alpha \in (0,1]. \qquad (25) \end{aligned}$$

The addition $\oplus$ introduced by Yager is given as follows. For $A = \{m_{ik}/(x_i, \mu_k)\}$ and $B = \{n_{ik}/(x_i, \mu_k)\}$,

$$A \oplus B = \{(m_{ik} + n_{ik})/(x_i, \mu_k)\}. \qquad (26)$$

Theorem 7. The commutativity of the addition with the α-cut holds:

$$(A \oplus B)_\alpha = A_\alpha \oplus B_\alpha. \qquad (27)$$

2.4 Fuzzy Bag Relations

We introduce crisp bag relations and fuzzy bag relations at a time. Let X, Y, and Z be three universal sets. A crisp bag relation P on $X \times Y$ is defined to be a crisp bag of $X \times Y$. Moreover a fuzzy bag relation R is defined to be a fuzzy bag of $X \times Y$, in other words, it is a crisp bag of $X \times Y \times I$. We define the grade sequence form of a fuzzy bag relation by the same way as for fuzzy bags. They are denoted by

$$\mu^j_R(x,y),\ j = 1, 2, ..., p,\ x \in X, y \in Y. \qquad (28)$$

The composition of two crisp bag relations P and Q is defined by

$$C_{P \circ Q}(x,z) = \bigvee_{y \in Y} (C_P(x,y) \wedge C_Q(y,z)). \qquad (29)$$

The composition of two fuzzy bag relations R and S is now defined:

(IX) [composition of fuzzy bag relations]

$$\mu^j_{R\circ S}(x,z) = \bigvee_{y\in Y} (\mu^j_R(x,y) \wedge \mu^j_S(y,z)),$$
$$j = 1, 2, ..., p, \ x \in X, z \in Z.$$

Theorem 8. The composition given in (IX) is well-defined. Namely,

$$\mu^1_{R\circ S}(x,z) \geq \mu^2_{R\circ S}(x,z) \geq ... \geq \mu^p_{R\circ S}(x,z) \qquad (30)$$

for all $x \in X$ and $z \in Z$.

Theorem 9. Let R and S be two fuzzy bag relations. Then the following commutativity between the composition and the α-cut holds.

$$(R \circ S)_\alpha = R_\alpha \circ S_\alpha. \qquad (31)$$

3. CONCLUSIONS

We have introduced new operations of fuzzy bags by the use of the grade sequence form. A key operation is the α-cut for fuzzy bags. The commutativity between an operation and the α-cut has been shown by the above theorems.

In this paper it is unnecessary to rearrange the order of the grade sequence after an operation. However, when considering other operations, we should consider reordering of the sequence. Examples of such operations are complement(s) and Cartesian product(s). Whether or not we should avoid such operations is an important subject of further studies.

Applications of fuzzy bags can be divided into two categories of applications for fuzzy sets and related theories and applications to information systems. Namely,

- Yager [11] proposes the use of crisp bags to the cardinality of fuzzy sets. Other theoretical applications are expected. For example, an application of fuzzy bags to rough sets is being considered [7].

- As noted in the introduction, crisp bags can be applied to the modeling of relational database systems. Hence fuzzy bags have applications to fuzzy relational database. New functions in an query language for fuzzy relational databases can be considered.

To summarize, the theory of fuzzy bags have just begun. There are many aspects to be studied further.

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