Proceedings of the 11th
KACC, October 1996

Stabilization and Trajectory Control of the Flexible Manipulator with Time-Varying Arm Length

°Chang-Yong Park*, Toshiro Ono*, and Yulwan Sung**

*Dep. of Mechanical Systems Engineering, College of Engineering,
Osaka Prefecture University, 1-1 Gakuen-cho, Sakai, Osaka 593, Japan
Tel: +81-722-52-1161(2259); Fax: +81-722-59-3340; E-mail: park@ctrlab.mecha.osakafu-u.ac.jp.
**Dep. of Computer and Systems Sciences, College of Engineering
Tel: +81-722-52-1161(2280); Fax: +81-722-59-3340; E-mail: sung@ctl.cs.osakafu-u.ac.jp.

Abstracts This paper deals with the flexible manipulator with rotaional and translational degrees of freedom, which has an arm of time-varying length with the prismatic joint. The tracking control problem of the flexible manipulator is considered. First we design the controller of the 2-type robust servo system based on the finite horizen optimal control theory for the trajectory planned as a discontinuous velocity. Next, to reduce the tracking error, we use the method of the dynamic programming and of modifying the reference trajectory in time coordinate.

The simulation results show that the dynamic modeling is adequate and that the asymptotic stabilization of the flexible manipulator is preserved in spite of nonlinear terms. The PTP control error has been reduced to zero completely, and the trajectory tracking errors are reduced sufficiently by the proposed control method.

Keywords Time-Varying Arm Length, Asymptotic Stability, L-type Robust Servo, Trajectory Control, Modified Trajectory

1. INTRODUCTION

The flexible manipulator with light weight, which has an arm of time-varying length, has the advantages of fast operation and the attainment of large workspace. But this specific type manipulator give rise to the concern about the elastic vibration suppression of the flexible manipulator which is critical to the tracking accuracy of robots. And also its nonlinear dynamics make it difficult to design the controller.

As the previous works, the modeling and the dynamic behavior of the flexible manipulator with a prismatic and rotational joints were derived and control problems were discussed [1] - [4]. Park et al. [1] achived the feedforward control by using the motion control based on the optimal control theory. The feedforward control strategy is important in reducing the tracking error of the end-effector of the flexible manipulator along the given trajectory. Yamada et al. [2] formulated the positioning control problem as the optimal regulator problem with conditions of making the load cease to oscillate at the desired position and of minimizing the input energy. Jiang et al. [3] discussed the existence of the solution of the trajectory tracking problem. Park et al. [4] discussed the modeling and tracking control problem of the flexible manipulator with the prismatic joint along the arc line trajectory. But, these works do not take the nonlinear terms into consideration.

In this work, we derived the dynamic model of the flexible manipulator with time-varying arm length. Although the dynamic model has highly nonlinear terms, there were little influences of the nonlinear terms on the behavior of the system when the operation range was small. Therefore, to achieve the asymptotic stability of the system, we used a linear time-varying feedback control law. The point-to-point(PTP) control could also be exactly achieved by using the 2-type robust servo controller in spite of nonlinear terms. For tracking along to the discontinuous-velocity trajectory, we designed the controller of the 2-type robust servo system based on the finite horizen optimal control theory. By using this controller and modifying the reference trajectory in time coordinate, we could reduce sufficiently the tracking errors to an adequate level.

2. DYNAMIC MODEL OF THE MANIPULATOR

2.1 Differential Equation of the Manipulator

We discuss the flexible manipulator shown in Fig. 1 which consists of two links, one revolute joint and one prismatic joint. Link 1 is a rigid manipulator which has total length $L_1 + L_3$, mass per unit length ρ_1 and area A_1. Link 2 is a flexible manipulator with variable arm length whose total length is L_2, mass per unit length ρ_2, area A_2, moment of inertia I_2 and Young modulus E_2. The origins of link 1 and link 2 are O_1 and O_2 respectively. Link 2 does translation motion about link 1, with a prismatic joint. The elastic part of link 2 is defined by the distance between the tips of link 1 and link 2. An ideal prismatic joint without any undesirable gap in all directions is assumed.

With respect to the inertial coordinate(O_1XY), the angular displacement of link 1 is represented by $\theta(t)$ (the counter-clock direction is set to be positive). The input commands, which are the torques to be applied at each actuator, are represented by $u(t)$ (the counter-clock direction is set to be positive). The line between the origin of link 1 and the tip of the end effector is called by a virtual rigid manipulator. $R_{vir}(t)$ denotes its radial length and $\theta_{vir}(t)$ denotes its angular displacement with respect to the inertial coordinate. $\bar{y}(\bar{x},t)$ denotes the deflection of the flexible manipulator at $\bar{x}$ which is a distance from the origin of the link 2, O_2. We assume that the elastic displacement of link 2, $\bar{y}(\bar{x},t)$, is very small compared to the rigid displacement.

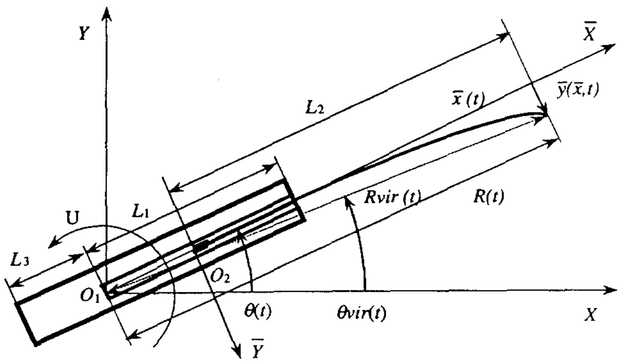

Fig. 1. Schematic diagram of the flexible manipulator with time-varying arm length.

We set some assumptions for modeling the flexible manipulator.

[**A**.1] Elastic deflection on the second link can be modeled as Euler-Bernoulli Beam.
[**A**.2] Radial displacement $R(t)$ is determined beforehand as follows.

$$R(t) = (t^2 - 2t + 4)^{\frac{1}{2}}$$

[**A**.3] Elastic deflection $\bar{y}(\bar{x},t) = 0$ at $\bar{x} = L_1 + L_2 - R(t)$.

Using the following relationship, we can nondimensionalize the elastic part of the flexible manipulator.

$$\bar{x} = (R(t) - L_1)x + (L_1 + L_2 - R(t)) \quad (1)$$

$$\bar{y}(\bar{x},t) = (R(t) - L_1)y(x,t) \quad (2)$$

From Euler-Lagrange equation and by nondimensionalizing, we obtain the nonlinear differential equations and its boundary conditions of the flexible manipulator as follows.

$$\begin{aligned}
&\frac{1}{3}[\rho_1 A_1(L_1^3 + L_3^3) + \rho_2 A_2(L_1^3 - (R - L_2)^3)]\ddot{\theta} \\
&+\rho_2 A_2 H \int_0^1 [(L_1 + Hx)(H\dot{y} + \dot{R}y)\dot{R} + H\dot{R}\dot{\theta} \\
&+(L_1 + Hx)(H\ddot{y} + \ddot{R}y - \ddot{\theta}) - H\ddot{R} - x\dot{R}\dot{\theta} \\
&+H^2\ddot{\theta}y^2 + 2y\dot{R}^2 + H\dot{y}\dot{R} + 2(L_1 + Hx)\dot{R}\dot{y} \\
&+2H^2 y\dot{y}\dot{\theta} + (H\dot{y} + \dot{R}y)\dot{R}x + 3H\dot{R}\dot{\theta}y^2] = u \quad (3)
\end{aligned}$$

$$\begin{aligned}
&\frac{EI}{H^3}\frac{\partial^4 y}{\partial x^4} + \rho_2 A_2[(1 - H^2)(\ddot{R}y + H\ddot{y}) \\
&+H^3 y\dot{\theta}^2 - (Hx + L_1)\ddot{\theta} - Hy\dot{R}^2 \\
&+(2H^2 x + L_1 H + H^2)\dot{R}\dot{\theta} - H^2\dot{y}\dot{R}] = 0 \quad (4)
\end{aligned}$$

$$y(0,t) = \frac{\partial y(0,t)}{\partial x} = \frac{\partial^2 y(1,t)}{\partial x^2} = \frac{\partial^3 y(1,t)}{\partial x^3} = 0 \quad (5)$$

where, $H = R(t) - L_1$.
To approximate the partial differential equations by ordinary differential equations, the method of modal analysis is used.

2.2 Method of Modal Analysis

We can use the method of modal analysis to solve the nonlinear partial differential equations, Eqs.(3) and (4). We assume the elastic deflection $y(x,t)$ as the linear combination of the product of proper modal function $\phi_i(x)$ and time function $q_i(t)$

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(x)q_i(t) \quad , \quad (i = 1, 2, \cdots). \quad (6)$$

But, this method is focused on the constant length links. Therefore we cannot separate variables for the link with time-varying arm length. So we derive the numerical model by using the following tricky way of thinking to make it possible to use the method of modal analysis

i) We get the equation of elastic deflection with constant length at a certain time t under the assumption that the elastic arm will oscillate in the same manner even after that time instant.
ii) After a tiny time δt has elapsed, we derive the equation of elastic deflection with constant length at that time instant under the similar pressumption that it will continue the same oscillation after that time.
iii) Repeat the same process, and bring a tiny time δt to tend to zero.

By using the method of modal analysis, the dynamics of the flexible manipulator with a prismatic joint are given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \delta f(x(t)) \quad (7)$$

$$x(t) = [\theta(t) \;\; \dot{\theta}(t) \;\; q(t) \;\; \dot{q}(t)]^T \quad (8)$$

$$\delta f(x(t)) = [0 \;\; \Delta_1 \;\; 0 \;\; \Delta_2]^T , \quad (9)$$

where, $t \in R$ is a time, $x(t) \in R^n$ are states, $u(t) \in R$ is a input, and $A(t), B(t)$ are the matrices as the functions of $R(t)$ and/or $\ddot{R}(t)$ which denotes 2nd time derivative of $R(t)$. $\delta f(x(t)) \in R^n$ are nonlinear terms and its entries are

$$\Delta_1 = -0.006546\rho_2 A_2 H^3\dot{\theta}^2 q_1 \quad (10)$$

$$\begin{aligned}
\Delta_2 = &\; 0.000415\rho_2 A_2 H^2 q_1 \\
&*[3\dot{R}\dot{\theta}q_1 + 2H\dot{\theta}\dot{q}_1 + H\ddot{\theta}q_1] . \quad (11)
\end{aligned}$$

3. STABILITY OF THE FLEXIBLE MANIPULATOR

We simulate the stability and the trajectory controls of the system obtained in the previous chapter. The parameters used in the simulations are : lengths of links L_1 = 1000 mm, L_2 = 1500 mm, L_3 = 500 mm, mass per unit length ρA = 0.504 kg/m, flexural rigidity of link 2, EI = 47.38 Nm^2 and final time t_f = 2 sec.
In case $\delta f(x(t)) = 0$ in Eq.(7), we obtain a feedback gain, $K = [1 \;\; 1 \;\; 8 \;\; 3]$ which stabilizes the time-varying nominal system, and put it in Eq.(7). Then Eq.(12) is obtained as follows.

$$\dot{x}(t) = (A(t) - B(t)K)x(t) + \delta f(x(t)) \quad (12)$$

Consider $V(x(t)) = x^T(t)Px(t)$ as a Lyapunov function candidate, where P is a positive definite matrix as follows.

$$P = \begin{bmatrix} 1.6592 & 0.4092 & 3.0205 & 0.0188 \\ 0.4092 & 0.3988 & 3.4794 & 0.0245 \\ 3.0205 & 3.4794 & 153.8709 & 1.6616 \\ 0.0188 & 0.0245 & 1.6616 & 0.0261 \end{bmatrix}$$

By taking the derivative of $V(x(t))$ with respect to time along the trajectory of Eq.(12), we obtain Eq.(13).

$$\dot{V}(x(t)) = -x^T(t)Q(t)x(t) + 2x^T(t)P\delta f(x(t)) \quad (13)$$

From Eq.(13), we can see that the condition of $\dot{V}(x) < 0$ is satisfied at $\| x_0 \| \leq 4.7$ for $Q = diag\{1.5 \; 1 \; 1 \; 1.3\}$. Therefore, the asymptotic stability of the system can be preserved at the neighborhood of the equilibrium point, $\| x(t) \| \leq 4.7$, in spite of the nonlinear terms. Thus, at the next chapter, we consider the tracking problem of the flexible manipulator in case that the operating region of manipulation satisfies the condition, i.e., $\| x(t) \| \leq 4.7$.

4. TRAJECTORY CONTROL OF THE FLEXIBLE MANIPULATOR

We consider to make the end-effector track uniformly along a straight line trajectory, by means of operating the two motors individually. The straight line trajectory is the trajectory of two seconds which has the discontinuous velocities at the initial point and the final point. The problem which makes the end-effector track along the given trajectory is the same as the problem which make $R(t)$ and $\theta(t)$ track along the desired radial displacement function $R_d(t)$ and the angular displacement function $\theta_d(t)$, respectively. Since it is assumed that the radial displacement function is given beforehand and the trajectory error in radial direction is very small compared to that in rotational direction, we discuss only the tracking control problem along the reference trajectory (i.e., the desired angular trajectory, $\theta_d(t)$).

There are some methods to obtain the control law for the trajectory tracking. Among those, we can determine the control input with feedback gain, K by means of the pole-assignment regulator. And also we can directly plan the dynamic characteristics of the system by using the pole-assignment regulator. But, it is possible that the input may increase immensely or that the system may be sensitive to variations of the parameters. Fig. 2 is the result of the simulation for the control input obtained, here the dashed line denotes the desired angular trajectory $\theta_d(t)$ and the solid line denotes the angular displacement of the virtual rigid manipulator $\theta_{vir}(t)$. Fig. 2 shows that the tracking error is great. From the results of Chapter 3, because there is little influence of the nonlinearity on the behavior of the system when the initial values are small, the asymptotic stability can be preserved though the linear time-varying feedback control law is used. Thus we designed the controller of the L-type robust servo system. Because the desired rotational angle (i.e., $\theta_d(t)$) is similar to the ramp function, we designed the controller of the 2-type robust servo system as follows.

$$\dot{z}_1 = r - y_d \quad (14)$$

$$\dot{z}_2 = z_1 \quad (15)$$

$$u = F_0 x + F_1 z_1 + F_2 z_2 \quad (16)$$

The response of the PTP control by the 2-type robust servo controller is shown in Fig. 3, and the dashed line denotes the desired angular trajectory, $\theta_d(t)$, and the solid line denotes the angular displacement of the virtual rigid manipulator $\theta_{vir}(t)$. From Fig. 3, the end-effector of moves exactly from the starting point to the destination point. As to the trajectory control, to reduce the tracking error, we formulate the control law as the finite horizen optimal control problem.

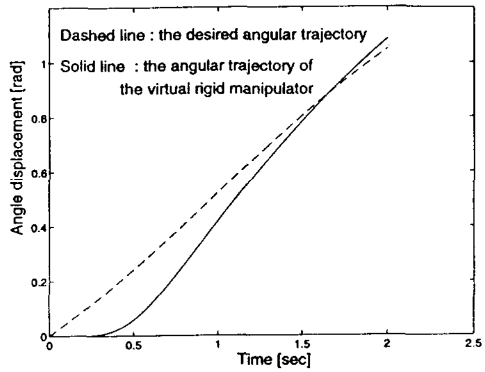

Fig. 2. Response due to the control law by the pole assignment regulator.

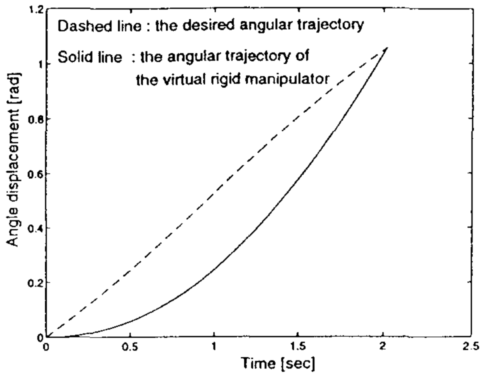

Fig. 3. PTP control of the end-effector by the 2-type robust servo controller.

We design the optimal state feedback gain $F(t)$ for the system, by defining the performance index J with $R > 0$, $Q > 0$ and $Q_f > 0$ as

$$J = x^T(t_f)Q_f x(t_f) + \int_0^{t_f} (x^T Qx + u^T Ru)d\tau, \quad (17)$$

where, matrices Q_f, Q and R have continuous entries, be symmetric, and be nonnegative and positive definite, $Q_f = diag(0.01 \; 0.01 \; 0.01 \; 0.01 \; 1 \; 1)$,$R = [1]$, and $Q = diag(1 \; 1 \; 10 \; 1 \; 60000 \; 60000)$ respectively.

Then the feedback gain $F(t)$ that minimizes the index J is given by

$$F(t) = -R^{-1}B^T P(t), \quad (18)$$

where, $P(t)$ is the positive definite solution of the Riccati equation,

$$-\dot{P} = Q + A^T P + PA - PBR^{-1}B^T P,$$
$$P(t_f) = Q_f. \quad (19)$$

Thus, the control input obtained by the dynamic progamming is as follows.

$$u(t) = F(t)x(t) \tag{20}$$

The response of the simulation for the above control input is shown in Fig. 4. Here, the dashed line denotes the desired angular trajectory, $\theta_d(t)$, and the solid line denotes the angular displacement of the virtual rigid manipulator $\theta_{vir}(t)$. As expected, the tracking error is reduced to some accuracy level. Nevertheless, this system cannot establish a perfect tracking because it is a non-minimum phase system.

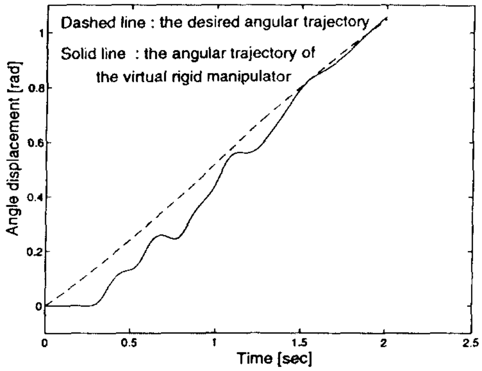

Fig. 4. Response due to the control law by the dynamic programming.

Thus, to reduce sufficiently the tracking errors, we scheme to modify the reference trajectory(i.e., the desired angular trajectory $\theta_d(t)$). The reference trajectory is of two seconds which has the discontinuous velocities at the initial point and the final point. To remove the discontinuous velocities, we can make the modified reference trajectory by adding residual times to the reference trajectory at the initial point and the final point. By using residual times, the modified reference trajectory has the continuous velocities at the initial point and the final point. After all, we made the modified reference trajectory by using the ramp function during the first two seconds, from 0.0 sec to 2.0 sec, and the reference trajectory $\theta_d(t)$ during the next two seconds, from 2.0 sec to 4.0 sec. Then, we can obtain the optimal control law for the modified reference trajectory. The response of the new input obtained is shown in Fig. 5. Here, the dashed line denotes the desired angular trajectory $\theta_d(t)$, (i.e., the modified reference trajectory) and the solid line denotes the angular displacement of the virtual rigid manipulator $\theta_{vir}(t)$.

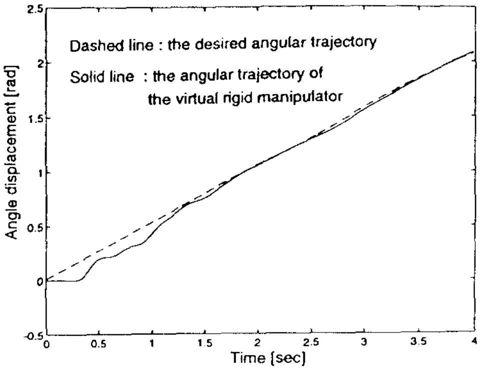

Fig. 5. Response due to the control law by the dynamic programming and the modified reference trajectory.

From Fig. 5, we can see that the tracking errors of the reference trajectory are considerly improved. TABLE 1 compares the results of this method with reference [1].

TABLE 1. Comparison of tracking errors of this method with the previous work.

Angular displacement [rad]	Reference [1]	Control law presented
Maximum error [rad]	0.0316	0.0125
Mean error [rad]	0.0148	0.0130

Here, we designed the controller of the 2-type robust servo system because the reference trajectory is similar to the ramp fuction. If the reference trajectory is the curve of a higher order of time, this controller obtained in the above cannot perform adequately. If the higher order-type robust servo controller is designed according to the order of the reference trajectory, the tracking error for any curved reference trajectory will be improved.

5. CONCLUSIONS

In this work, we derived the motion equations of the flexible manipulator which has the time-varying arm length, taking nonlinear terms into consideration. There were little influences of the nonlinear terms on the behavior of the system when the initial value is small. Therefore, the asymptotic stability of the system could be preserved though the linear time-varying feedback control law was used. By using the 2-type robust servo controller, the PTP control could also be exactly achieved in spite of nonlinear terms. When the discontinuous-velocity trajectory was given, by means of the controller of the 2-type robust servo system and the modified reference trajectory in time coordinate, we could reduce the tracking errors to an adequate level.

REFERENCES

[1] C.Y.Park and T.Ono, "Trajectory Control of a Flexible Manipulator with a Prismatic Joint," *Proceedings of '92 Korea Automatic Control Conference*, pp.613-617, Seoul, Korea, 1992.

[2] I.Yamada and M.Nakagawa, "Positioning Control of a Servomotor Mechanism with an Oscillatory Load," *Trans. of the Society of Instrument and Control Engineers*, Vol.18, No.1, pp.84-91, 1982.

[3] Z.H.Jiang and M.Uchiyama, "Compensability of Flexible Robot Arms," *Jour. of RSJ*, Vol.6, No.5, pp.416-423, 1988.

[4] C.Y.Park and T.Ono, "Trajectory Control of a Flexible Manipulator with a Prismatic Joint(Part 2.Sensitivity Evaluation in Case of a Circular Arc Trajectory)," *Bulletin of Osaka Prefecture Univ.*, Series A, Vol.42, No.1, pp.1-12, 1993.