# 분산 저차 구조의 $H_\infty$ 제어기 설계를 위한 시스템의 간략화 방법

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# Simplifying method for the design of decentralized reduced order $H_\infty$ controllers

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## Abstract

The simplifying method for the design of decentralized reduced order $H_\infty$ controller is considered in this paper. When the controller is reconstructed for the original system, the decentralized condition of the controller for the transformed system is generally destroyed with older simplifying method. In designing the decentralized controller, direct output feedthrough terms give some difficulties by using other station's input information. We proposed a new solution for this problem.

keywords : $H_\infty$ stabilization, Decentralization, Simplifying

## 1 Introduction

In this research, we consider the simplifying method of the general plant for the design of decentralized reduced-order $H_\infty$ controllers. Individual system has a reduced order observer. With these observers, the estimate of the states or worst case exogeneous inputs of a channel can be obtained. Then we can design the controller minimizing the $H_\infty$ norm of the transfer function matrix from exogeneous inputs to the controlled outputs.

The standard assumptions of the system is modified for the decentralized controller design. General plant includes a direct feedthrough term from input to the output, and controlled output has a direct feedthrough term from exogeneous input. Hence, past simplifying method to the assumptions for the general plant is not usable to the system with decentralized controller structure. We will start with the definition of the problem.
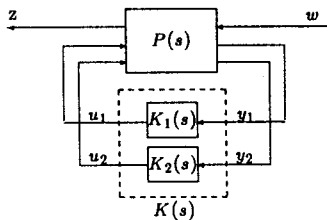


Figure 1: Design Structure

## 2 Preliminaries

Firstly, we consider the system having the $D_{11}$ and $D_{22}$ for the general plant.

$$P(s) \equiv \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases} \qquad (1)$$

We can rewrite above as a following matrix form :

$$\begin{bmatrix} \dot{x} \\ z \\ y_1 \\ y_2 \end{bmatrix} = \left[ \begin{array}{c|ccc} A & B_1 & B_2^1 & B_2^2 \\ \hline C_1 & D_{11} & D_{12}^1 & D_{12}^2 \\ C_2^1 & D_{21}^1 & D_{22}^1 & D_{22}^2 \\ C_2^2 & D_{21}^2 & D_{22}^3 & D_{22}^4 \end{array} \right] \begin{bmatrix} x \\ w \\ u_1 \\ u_2 \end{bmatrix} \qquad (2)$$

We want to design the decentralized controller $K(s)$ for the system $P(s)$.

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} K_1(s) & 0 \\ 0 & K_2(s) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad (3)$$

**Definition 1** *The system is said to have a decentralized fixed mode* $\lambda \in \mathbf{C}$ *with respect to* $\Phi$ *if*

$$\lambda = \bigcap_{K \in \Phi} sp[A + B_2 K (I - D_{22} K)^{-1} C_2]$$

*where* $sp(\cdot)$ *denotes the set of eigenvalues of* $(\cdot)$.

### 2.1 Assumptions

The above system must satisfy the following assumptions. Singular problem is not considered in this paper.

**A1.** there are no unstable decentralized fixed modes.
**A2.** $D_{12}$ and $D_{21}$ has full column/row rank, respectively.
**A3.** ( $A$   $B_2$   $C_2$ ) is stabilizable and detectable.

### 2.2 Simplifying The Assumptions

In this section, we try to remove the terms of $D_{11}$ and $D_{22}$ . For this, general procedure for the decentralized controller design is suggested by modifying the general simplying procedure.

Step 1 : Decentralized norm minimization problem

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$$

where $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$ .

By applying above transformation, we can obtain following LFT :

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[ \begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array} \right] \begin{bmatrix} x \\ w \\ \tilde{u} \end{bmatrix} \qquad (4)$$

where

$$\begin{aligned} \bar{A} &= A + B_2 F (I - D_{22} F)^{-1} C_2 \\ \bar{B}_1 &= B_1 + B_2 F (I - D_{22} F)^{-1} D_{21} \\ \bar{B}_2 &= B_2 (I - D_{22} F)^{-1} \\ \bar{C}_1 &= C_1 + D_{12} F (I - D_{22} F)^{-1} C_2 \\ \bar{C}_2 &= (I - D_{22} F)^{-1} C_2 \\ \bar{D}_{11} &= D_{11} + D_{12} F (I - D_{22} F)^{-1} D_{21} \\ \bar{D}_{12} &= D_{12} (I - D_{22} F)^{-1} \\ \bar{D}_{21} &= (I - D_{22} F)^{-1} D_{21} \\ \bar{D}_{22} &= (I - D_{22} F)^{-1} D_{22} \end{aligned}$$

**Step 2 :** Making $D_{11} = 0$

In real situation, it is generally not satisfied by the frequency weighted function for the closed loop shaping. Hence, this preocedure is needed for simplicity. Currently, some loop transformation is used, but it is not satisfactory for the system having decentralized structure. For the decentralized system, following transformation can only be used.

If we let $\bar{D}_{11} = D_{11} + D_{12} F (I - D_{22} F)^{-1} D_{21}$, following transformation will remove $\bar{D}_{11}$.

A. Find the decentralized form of matrix $F$ such that $\min_F \| \bar{D} \|$ $\gamma_o$.

B. Choose $\gamma > \gamma_o$ and performing next transformation.

$$\begin{aligned} \begin{bmatrix} \tilde{z} \\ w \end{bmatrix} &= \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ z \end{bmatrix} \\ &= \gamma^{-1} \begin{bmatrix} \gamma^{-1} \bar{D}_{11} & (I - \gamma^{-2} \bar{D}_{11} \bar{D}_{11}^*) \\ (I - \gamma^{-2} \bar{D}_{11}^* \bar{D}_{11}) & \gamma^{-1} \bar{D}_{11}^* \end{bmatrix} \begin{bmatrix} \tilde{w} \\ z \end{bmatrix} \end{aligned} \quad (5)$$

where $\Theta \Theta^* = \gamma^{-2} I$, $\| \Theta_{22} \|_2 < \gamma^{-2}, \forall \gamma > \gamma$.

By applying above transformation, we can obtain following LFT :

$$\begin{bmatrix} \dot{x} \\ \tilde{z} \\ y \end{bmatrix} = \left[ \begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \hat{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \hat{D}_{22} \end{array} \right] \begin{bmatrix} x \\ \tilde{w} \\ \tilde{u} \end{bmatrix} \qquad (6)$$

$$\begin{aligned} \hat{A} &= \bar{A} + \bar{B}_1 (I - \bar{D}_{11} \Theta_{22})^{-1} \tilde{C} \\ \hat{B}_1 &= \bar{B}_1 (I - \Theta_{22} \bar{D}_{11})^{-1} \Theta_{21} \\ \hat{B}_2 &= \bar{B}_2 + \bar{B}_1 \Theta_{22} (I - \bar{D}_{11} \Theta_{22})^{-1} \bar{D}_{12} \\ \hat{C}_1 &= \Theta_{12} (I - \bar{D}_{11} \Theta_{22})^{-1} \tilde{C}_1 \\ \hat{D}_{12} &= \Theta_{12} (I - \bar{D}_{11} \Theta_{22})^{-1} \bar{D}_{12} \\ \tilde{C}_2 &= \bar{C}_2 + \bar{D}_{21} \Theta_{22} (I - \bar{D}_{11} \Theta_{22})^{-1} \tilde{C}_1 \\ \tilde{D}_{21} &= \bar{D}_{21} (I - \bar{\Theta}_{22} \bar{D}_{11})^{-1} \Theta_{21} \\ \hat{D}_{22} &= \bar{D}_{22} + \bar{D}_{21} \Theta_{22} (I - \bar{D}_{11} \Theta_{22})^{-1} \bar{D}_{12} \end{aligned}$$

**Step 3 :** Making $D_{22} = 0$

Some feedforward term is added after designing controller. Several consideration is needed.

$$\begin{bmatrix} \dot{x} \\ \tilde{z} \\ y \end{bmatrix} = \left[ \begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \hat{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \hat{D}_{22} \end{array} \right] \begin{bmatrix} x \\ \tilde{w} \\ \tilde{u} \end{bmatrix} \qquad (7)$$

We apply following transformation.

$$\begin{aligned} \tilde{y} &= y - \hat{D}_{22} \tilde{u} \\ &= \tilde{C}_2 x + \tilde{D}_{21} \tilde{u} \end{aligned} \qquad (8)$$

Then the resulting tranfer system is as this :

$$\begin{bmatrix} \dot{x} \\ \tilde{z} \\ \tilde{y} \end{bmatrix}' = \left[ \begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \hat{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & 0 \end{array} \right] \begin{bmatrix} x \\ \tilde{w} \\ \tilde{u} \end{bmatrix} \qquad (9)$$

Now, our objective is complete. But one problem remains still unsolved. If you would like to reconstruct the controller for your original plant, other station's inputs are needed in subsystems by the terms $\hat{D}_{22}^2$ and $\hat{D}_{22}^3$, i.e., antidiagonal terms of $\hat{D}_{22}$ on above setting. Hence, following approximation is needed for decentralizing controllers.

$$\begin{aligned} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} \hat{D}_{22}^1 & \hat{D}_{22}^2 \\ \hat{D}_{22}^3 & \hat{D}_{22}^4 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \\ &\cong \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} \hat{D}_{22}^1 & 0 \\ 0 & \hat{D}_{22}^4 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} - \begin{bmatrix} 0 & \hat{D}_{22}^2 \\ \hat{D}_{22}^3 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1^e \\ \tilde{u}_2^e \end{bmatrix} \end{aligned}$$

Here, $\tilde{u}_i^e$ is the estimate of the other system's input. It can be estimated by some state estimates and replacement. Reduced-order observers gives some estimates of the system state, and from this, we can estimate the other station's inputs. Control strtrategy which is used by full state feedback system can be used. These method is used in the reference [1]. These method is good for the sytems without exchanging information each other. Hence, the assumption $\hat{D}_{22} = 0$ is generalized.

We can construct the simplified procedure for the general system with the above assumption without loss of generality. After now on, we only consider the decentralized reduced order $H_\infty$ controller for the simplified system. Theorem of the following section will justify our transformations.

## 2.3 Main Result

Now, we must show that the loop transformation does not destroy the representation of the decentralized controllers and that the order of the designed $H_\infty$ controller is not higher than before the transformation. Note that the structure of the controller as Figure 2. Let the state space realizations as this :

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2^e \end{bmatrix} = \tilde{K}_1(s) \tilde{y}_1, \qquad \begin{bmatrix} \tilde{u}_2 \\ \tilde{u}_1^e \end{bmatrix} = \tilde{K}_2(s) \tilde{y}_2 \qquad (10)$$

where

$$\tilde{K}_1(s) \equiv \left[ \begin{array}{c|c} \tilde{A}_{k1} & \tilde{B}_{k1} \\ \hline \tilde{C}_{k1} & \tilde{D}_{k1} \\ \tilde{C}_{k2}^e & \tilde{D}_{k2}^e \end{array} \right] = \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} \qquad (11)$$
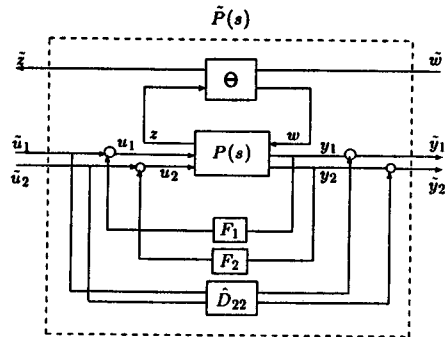


Figure 2: Loop Transformation

$$\tilde{K}_2(s) \equiv \left[ \begin{array}{c|c} \tilde{A}_{k2} & \tilde{B}_{k2} \\ \hline \tilde{C}_{k2} & \tilde{D}_{k2} \\ \tilde{C}_{k1}^e & \tilde{D}_{k1}^e \end{array} \right] = \begin{bmatrix} A_3(s) \\ A_4(s) \end{bmatrix} \qquad (12)$$

Let the realization for the controller $K_1(s)$ and $K_2(s)$ as this :

$$u_1 = K_1(s) \, y_1, \quad u_2 = K_2(s) \, y_2 \qquad (13)$$

where

$$K_1(s) \equiv \left[ \begin{array}{c|c} A_{k1} & B_{k1} \\ \hline C_{k1} & D_{k1} \end{array} \right], \quad K_2(s) \equiv \left[ \begin{array}{c|c} A_{k2} & B_{k2} \\ \hline C_{k2} & D_{k2} \end{array} \right] \qquad (14)$$

With above and equation (3) and (10), we can conclude as this lemma.

**Theorem 1 (Reconstruction)** *If we can design the decentralized controller $\tilde{K}(s)$ for the systems for $\tilde{P}(s)$, then we can reconstruct the controller $K(s)$ for the system $P(s)$ having no higher oreder realization.*

$$u_1 = \left[ F_1 + \left\{ I + A_1(s)(I + \hat{D}_{22}^2 A_2(s))^{-1} \hat{D}_{22}^1 \right\}^{-1} \right. \\ \left. \left\{ A_1(s)(I + \hat{D}_{22}^2 A_2(s))^{-1} \right\} \right] y_1 \qquad (15)$$

$$u_2 = \left[ F_2 + \left\{ I + A_3(s)(I + \hat{D}_{22}^3 A_4(s))^{-1} \hat{D}_{22}^4 \right\}^{-1} \right. \\ \left. \left\{ A_3(s)(I + \hat{D}_{22}^4 A_4(s))^{-1} \right\} \right] y_2 \qquad (16)$$
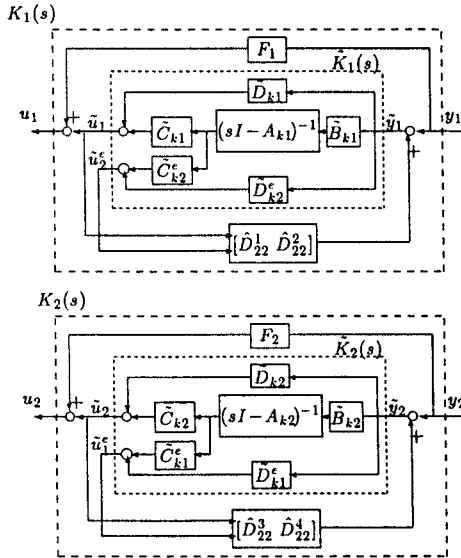


Figure 3: Reconstruction of Decentralized Controller for $K(s)$ with $\tilde{K}(s)$

## 2.4 Controller design formulation for the simplified system

Now after, we can assume that $D_{11}$ and $D_{22}$ equal zero without loss of generality.

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \qquad (17)$$

We can rewrite above as a following matrix form :

$$\left[ \begin{array}{c} \dot{x} \\ \hline z \\ y_1 \\ y_2 \end{array} \right] = \left[ \begin{array}{c|ccc} A & B_1 & B_2^1 & B_2^2 \\ \hline C_1 & 0 & D_{12}^1 & D_{12}^2 \\ C_2^1 & D_{21}^1 & 0 & 0 \\ C_2^2 & D_{21}^2 & 0 & 0 \end{array} \right] \left[ \begin{array}{c} x \\ \hline w \\ u_1 \\ u_2 \end{array} \right] \qquad (18)$$

For above system, we would like to design the controller having fixed structure :

$$u = \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{cc} K_1(s) & 0 \\ 0 & K_2(s) \end{array} \right] \left[ \begin{array}{c} y_1 \\ y_2 \end{array} \right] \qquad (19)$$

We can write above equation to following forms :

$$\tilde{K}_1(s) \equiv \begin{cases} \dot{\xi}_1 = F_1 \xi_1 + G_1 y_1 + [H_{11} \; H_{12}] \left[ \begin{array}{c} u_1 \\ u_2^\epsilon \end{array} \right] \\ u_1 = L_1 \xi_1 + M_1 y_1 \\ u_2^\epsilon = L_2^\epsilon \xi_1 + M_2^\epsilon y_1 \end{cases} \qquad (20)$$

$$\tilde{K}_2(s) \equiv \begin{cases} \dot{\xi}_2 = F_2 \xi_2 + G_2 y_2 + [H_{21} \; H_{22}] \left[ \begin{array}{c} u_1^\epsilon \\ u_2 \end{array} \right] \\ u_1^\epsilon = L_1^\epsilon \xi_2 + M_1^\epsilon y_2 \\ u_2 = L_2 \xi_2 + M_2 y_2 \end{cases} \qquad (21)$$

## 3 Conclusion

The remained problem is the construction of reduced-order observers. By [3] and above equations, we will try to connect the decentralization and reduced-order observer design problem. Bounded real lemma will help this links.

Our works are still constructing. We show that the simplifying method for the controller having fixed structure can not be applied to design the controller with the fixed structures. Some other simplifying method is needed and we provide it. Hence, the problem in the decentralized controller design was made easy to deal with. Reduced-order consroller design problem can be solved by manipulating the related equations.

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