Optimized Compact Finite Difference Schemes for Computational Aeroacoustics

수치적 공력음향해석을 위한 집적유한차분법의 개발

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Abstract

비정상(unsteady) 압축성(compressible) 유동에 의한 공력음향(aeroacoustics)을 모사하여 공력소음원을 해석하기 위해서는 고차(high order)의 정확도와 높은 해상도(resolution)를 가지며, 상대적으로 계산시간을 많이 필요로 하지 않는 외재적(explicit) 유한차분법이 필수적으로 요구된다. 이것은 주어진 차분방식과 격자계로써 공간과 시간상에 존재하는 미소크기의 파동성분들을 충분히 구현하여야 만족할 만한 수치해를 얻을 수 있기 때문이다. 본 연구에서는, 그러한 유한차분법 중 최근에 관심의 대상이 되고있는 삼각(tridiagonal) 또는 오각(pentadiagonal) 집적유한차분법(compact finite difference scheme)이 최대의 해상도를 갖도록 하는 수학적인 방법을 개발하고, 이 방법으로써 새롭게 집적유한차분법을 최적화 하였다. 개발된 최적화 방법은, 푸리에 해석법(Fourier analysis)을 통하여 파동수(wavenumber) 영역에서 수학적으로 계산된 위상오차(phase error)를 최소화 하는 것이며, 이러한 개념과 방법은 본 연구에서 처음으로 집적유한차분법에 적용되었다. 여러가지 절단정확도(truncation order)에 대해서 최적화 된 집적유한차분법들이 실제 공간과 시간상에서 보여주는 정확도와 오차특성을 알아보기 위하여, 이 방법들을 1차원 선형파동방정식에 적용하였고, 이 결과를 통하여 가장 정확하고 효과적인 절단정확도의 집적유한차분법을 선별하였다. 특히, 오각(pentadiagonal)법에 비해 더욱 효율적인 6차 삼각(tridiagonal)법을 1차원 Euler방정식에 적용하여, 비선형 파동에 대한 모사를 수행할 수 있었다.

1. Introduction

Recently, the necessities of a finite difference scheme which has high spatial and temporal order of truncation have been increased for direct numerical simulations(DNS) [1]-[4] and computational aeroacoustics (CAA) [5]-[6], because the problems of DNS and CAA would contain high frequency(short wave length) and small amplitude wave components. In addition to the high truncation order, the resolution of a scheme has been emphasized in that it will determine the number of grid points per wave length required to resolve the shortest wave component in the actual computation with the scheme. A scheme can be more accurate actually when its resolution characteristics are improved, and this was shown by Tam[7] and Lele[8].

The main objective of this paper is an analytic optimization of the compact finite difference schemes for the purpose of using on problems of a wide range of spatial length scales. This paper shows that an analytic optimization produces the maximum spatial resolution characteristics of the compact finite difference approximation in evaluation of the spatial first derivatives. A method for the analytic optimization is to minimize the dispersive errors in wavenumber domain. In this paper, it is further developed for the first application to the compact - not unidiagonal but multidiagonal - schemes.

Both the resolution and the truncation order of a scheme determines the overall error characteristics of its finite difference approximation in actual computations. And the actual error characteristics depends on the scheme itself, i.e. whether tridiagonal or pentadiagonal. Thus, it is not easy to see the overall performances of the optimized compact tridiagonal and pentadiagonal schemes of various truncation orders until some numerical simulations are followed to visualize their actual error characteristics and compare their actual accuracy. The paper presents the comparisons between the optimized compact schemes with a variety of truncation orders for tridiagonal and pentadiagonal cases, respectively. The comparisons are made by the numerical simulations of simple linear wave convection, and the most effective tridiagonal and pentadiagonal schemes are found. And with these comparisons, the usefulness of an optimized tridiagonal compact scheme which is more efficient than a pentadiagonal one is discussed and verified by its applications to nonlinear waves.

2. Compact Discretization

The schemes presented here are the generalizations of Padé scheme (see Ref. 8,9,10 and 11) of 7 point stencil as below :

$$\begin{aligned} &\beta f'_{i-2} + \alpha f'_{i-1} + f'_i + \alpha f'_{i+1} + \beta f'_{i+2} \\ &= c\frac{f_{i+3} - f_{i-3}}{6\Delta x} + b\frac{f_{i+2} - f_{i-2}}{4\Delta x} + a\frac{f_{i+1} - f_{i-1}}{2\Delta x} \end{aligned} \tag{1}$$

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The relations between the coefficients a,b,c,α and β are derived by matching the Taylor series coefficients of various orders. These relations are (see also Ref. 8)

$$a + b + c = 1 + 2\alpha + 2\beta \quad : \text{ 2nd order} \tag{2}$$

$$a + 2^2 b + 3^2 c = 2\frac{3!}{2!}(\alpha + 2^2\beta) \quad : \text{ 4th order} \tag{3}$$

$$a + 2^4 b + 3^4 c = 2\frac{5!}{4!}(\alpha + 2^4\beta) \quad : \text{ 6th order} \tag{4}$$

$$a + 2^6 b + 3^6 c = 2\frac{7!}{6!}(\alpha + 2^6\beta) \quad : \text{ 8th order} \tag{5}$$

$$a + 2^8 b + 3^8 c = 2\frac{9!}{8!}(\alpha + 2^8\beta) \quad : \text{ 10th order} \tag{6}$$

Only the 8th order tridiagonal scheme($\beta = 0$) and the 10th order pentadiagonal scheme($\beta \neq 0$) which are of the highest order have the unique coefficients. The other lower order schemes should have free coefficients that are not determined until some more constraints are imposed and these constraints can improve the resolution characteristics. In this paper, the analytic and systematic constraints for determination of the free coefficients are considered. This kind of constraints are the minimization of dispersive(phase) errors in wavenumber domain, i.e. the stencil wavenumber optimization introduced by Tam and Webb[7].

3. Fourier Analysis of Errors

The finite difference equation (1) is of a central difference, thus it has no dissipative errors. In this section the differencing errors of Eq. (1) are analyzed in terms of the dispersive(phase) errors. A Fourier analysis provides an effective way to quantify the dispersive errors and resolution characteristics of a differencing approximation, so this quantification would be used further to guide an analytic optimization of the differencing scheme. The finite difference equation that Tam and Webb[7] used for the dispersion-relation-preserving scheme is a unidiagonal($\alpha = \beta = 0$) case of Eq. (1), i.e. it is not a compact(multidiagonal) scheme. It is reasonable to expand Tam's finite difference equation to Eq. (1) for the purpose of increasing the truncation order and resolution. The Fourier transform of the left and right sides of Eq.(1) is

$$\begin{aligned} & i\omega(\beta e^{-2i\omega\Delta x} + \alpha e^{-i\omega\Delta x} + 1 + \alpha e^{i\omega\Delta x} + \beta e^{2i\omega\Delta x})\tilde{f} \quad = \\ & \left[\frac{c}{6\Delta x}(e^{3i\omega\Delta x} - e^{-3i\omega\Delta x}) + \frac{b}{4\Delta x}(e^{2i\omega\Delta x} - e^{-2i\omega\Delta x}) + \frac{a}{2\Delta x}(e^{i\omega\Delta x} - e^{-i\omega\Delta x})\right]\tilde{f} \end{aligned} \tag{7}$$

From this equation, the scaled modified wavenumber is derived as (see also Ref. 8)

$$\overline{\omega}\Delta x \equiv \frac{a\sin(\omega\Delta x) + \frac{b}{2}\sin(2\omega\Delta x) + \frac{c}{3}\sin(3\omega\Delta x)}{1 + 2\alpha\cos(\omega\Delta x) + 2\beta\cos(2\omega\Delta x)} \tag{8}$$

where $\overline{\omega}$ is a modified wavenumber. The modified wavenumber, defined as Eq. (8), is used for the analysis of finite difference errors. To assure that the Fourier transform of the finite difference scheme is a good approximation of that of partial derivative, the modified wavenumber should coincide with the corresponding true wavenumber ($\overline{\omega} = \omega$) in the wide range of wavenumbers(i.e. $0 \leq \omega\Delta x < \pi$) as possible.

4. Optimization of Coefficients

An integrated error(weighted deviation) is newly defined in this paper (see also Ref. 7) as

$$E \equiv \int_0^{r\pi} (\omega\Delta x - \overline{\omega}\Delta x)^2 W(\omega\Delta x) d(\omega\Delta x) \tag{9}$$

where '$W(\omega\Delta x)$' is a weighting function, and the character 'r' is a factor to determine the optimizing range($0 < r \leq 1$) under consideration. The integrated error defined as Eq. (9) is different from that of Tam and Webb[7] in that it contains the weighting function and the range factor r (see Ref. 7 and 14). The weighting function in the Eq. (9) would make the equation analytically integrable and the integrand be weighted near the high wavenumber range(near π) where much dispersive error exists. Then the integrated error for this scheme is expressed as

$$E = \int_0^{r\pi} \left(\kappa - \frac{\frac{c}{3}\sin(3\kappa) + \frac{b}{2}\sin(2\kappa) + a\sin(\kappa)}{1 + 2\alpha\cos(\kappa) + 2\beta\cos(2\kappa)} \right)^2 W(\kappa)\, d\kappa \tag{10}$$

where $\kappa = \omega\Delta x$. The integrated error in Eq. (10) is a function of the coefficients a,b,c,α, and β. It is necessary to find the optimum values of the coefficients that would minimize the integrated error. The conditions that make E be a local minimum value are proposed by Tam and Webb[7] as follows :

$$\frac{\partial E}{\partial a} = 0 \quad (12), \quad \frac{\partial E}{\partial b} = 0 \quad (13), \quad \frac{\partial E}{\partial c} = 0 \quad (14), \quad \frac{\partial E}{\partial \beta} = 0 \quad (15), \quad \frac{\partial E}{\partial \alpha} = 0 \quad (16)$$

The equations (2)-(6) and the equations (12)-(16) provide a system of linear algebraic equations by which the coefficients can be determined. The equations (12)-(16) which can minimize the integrated error defined as Eq. (11) provide equations to obtain the optimum coefficients in addition to the Eq. (2)-(6) derived by matching the Taylor series coefficients of various truncation orders.

In this paper, the weighting function is proposed to make the Eq. (10) integrable and also the integrand in Eq. (10) be weighted near the high wavenumber range close to π, by multiplying an exponential term as follow :

$$W(\kappa) = [\,(\,1 + 2\alpha\cos(\kappa) + 2\beta\cos(2\kappa)\,)\,e^{\kappa}\,]^2 \qquad (17)$$

Remind that the unidiagonal central scheme used in Ref. 7 do not need such a function, because $\alpha = \beta = 0$ which is no problem in integrating E analytically. The weighting function defined as Eq. (17) is devised to integrate the E analytically and decrease the dispersive errors in the high wavenumber range sufficiently because far more errors exist in the high wavenumber range than in the low wavenumber range.

5. Ranges of Optimization

The optimizing range factors of each scheme should be adjusted in order to get maximum resolution characteristics. These are so determined that the overshoot - the largest deviation of the modified wavenumber from the corresponding true wavenumber - is below 0.5% of the corresponding true wavenumber (see Ref. 12).

1. Tridiagonal ($\beta = 0$) :
   2nd order : $r = 0.820$, 4th order : $r = 0.790$, 6th order : $r = 0.715$.

2. Pentadiagonal ($\beta \neq 0$) :
   2nd order : $r = 0.90$, 4th order : $r = 0.890$, 6th order : $r = 0.865$, 8th order : $r = 0.815$.

The coefficients obtained by using each of the above optimizing range factors and the weighting function of Eq. (17) for each scheme are presented in Table 1 and Table 2 for the tridiagonal schemes and the pentadiagonal schemes, respectively. The maximum resolution characteristics of the optimized compact schemes for various trunction orders are compared with those of the other standard finite difference schemes in Fig. 1.

6. Applications to Simple Waves

Linear Waves

This section illustrates an application of the optimized schemes developed in this paper to an initial value problem of 1-D linear wave equation. Convection of a simple linear wave is numerically simulated by the optimized compact schemes of various truncation orders in the evaluation of spatial derivatives. The 4th order Runge-Kutta method is used for temporal integration. The results with a given number of grid points for an initial wave shape are presented in Fig. 2-3. In Fig. 1, it is noticed that a lower order optimized compact scheme has more resolution in wavenumber domain than a higher order one, but it is expected that the lower scheme may produce more truncation errors, i.e. actual errors in space than the higher one. Both the resolution in wavenumber domain and the truncation order in space of a scheme determines the overall error characteristics of its finite difference approximation in actual computations. Also, the actual error characteristics depends on the scheme itself, i.e. tridiagonal or pentadiagonal. Thus, it is not easy to see the overall performances of the optimized compact schemes until some numerical simulations are followed to visualize their actual error characteristics and compare their actual accuracy as shown in Fig. 2-3.

One can notice that the 6th order optimized tridiagonal scheme and the 4th order optimized pentadiagonal scheme provide solutions which have less dispersive errors and thus retain the initial shape of a wave longer than the other compact schemes do. In other words, these have the best combinations of the resolution characteristics and the order of truncation, so these seem to be the most effective compact schemes. And especially, the 6th order optimized tridiagonal scheme is presented as an efficient and economic one because it has the overall performance comparable to the optimized pentadiagonal schemes, although it needs only tridiagonal matrix solver which is easy and fast for computation.

Nonlinear Waves

In this section, the application to an initial value problem of 1-D Euler equation is presented. Convection of a simple nonlinear wave is numerically simulated by the 6th order optimized tridiagonal scheme in the evaluation of spatial derivatives. The 4th order Runge-Kutta method is used for temporal integration.

The compact schemes optimized in this paper are of central differences, so their applications to the problems with high discontinuities produce noisy oscillations. In this nonlinear problem, numerical viscosity terms is needed to remove the spurious oscillations produced by high discontinuities. For example, Tam's recent implementations of the dispersion-relation-preserving scheme use an adaptive filter[5] which can damp out the spurious oscillations confined to a narrow range of high wavenumbers. Addition of these filter terms to the optimized compact schemes gives a solution possibly with much less oscillations near discontinuities. The results of application of the adaptive filter to the nonlinear wave are presented in Fig. 4 and Fig. 5. It is shown that the 6th order optimized tridiagonal scheme with the adaptive filter gives a satisfactory solution for the nonlinear problem.

7. Conclusions

In this paper, the high order compact finite difference schemes are optimized analytically to achieve the maximum resolution characteristics. The analytic optimization method and procedure suggested here for the multidiagonal schemes would be also useful to the other spatial or temporal finite difference schemes for the purpose of obtaining high resolution characteristics. The 6th order tridiagonal and 4th order pentadiagonal schemes are successfully optimized to be the most effective and accurate compact schemes in the actual computations. Especially, the usefulness of the optimized 6th order tridiagonal compact scheme which is very efficient compared with the pentadiagonal ones is verified well in this paper. It is pointed that the overall error characteristics of an optimized scheme in actual computation are dependent on the truncation order, the resolution and the scheme itself(tridiagonal or pentadiagonal).

8. References

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Tables

Table 1. Optimized coefficients for maximum resolution characteristics

Tridiagonal	a	b	c	α	β
2nd order	1.545790417	0.434249728	-0.078236437	0.450901855	0
4th order	1.551941906	0.361328195	-0.042907397	0.435181352	0
6th order	1.568098212	0.271657107	-0.022576781	0.408589269	0

Table 2. Optimized coefficients for maximum resolution characteristics

Pentadiagonal	a	b	c	α	β
2nd order	1.265667929	1.079904285	0.053798648	0.596631925	0.103053504
4th order	1.280440844	1.049309076	0.044465832	0.589595521	0.097512355
6th order	1.323482375	0.944394243	0.027596356	0.566458285	0.081278202
8th order	1.373189728	0.814447053	0.016707870	0.537265947	0.064906379

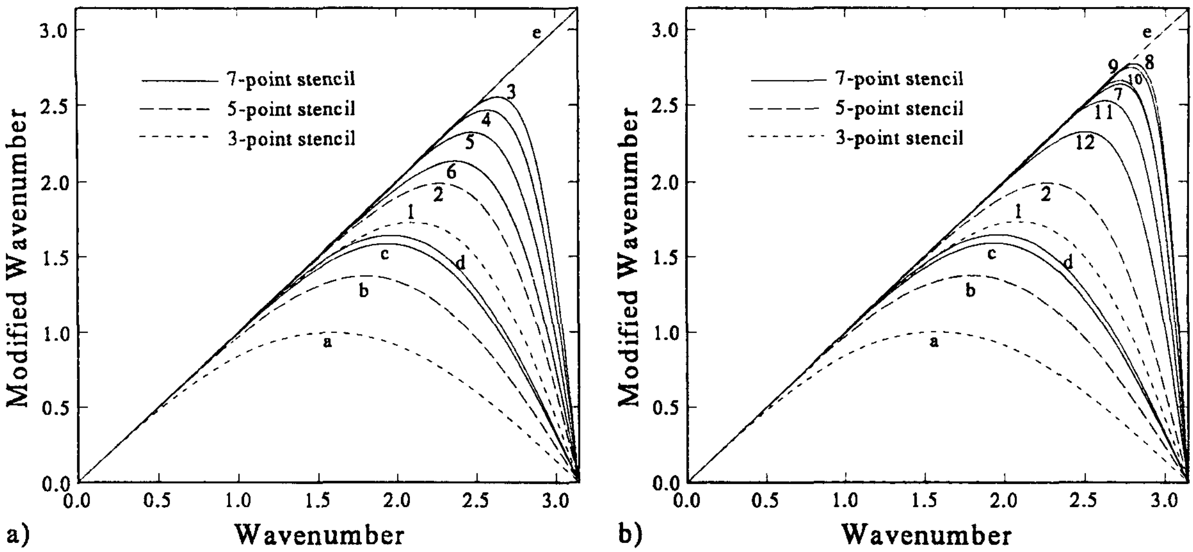

Fig.1 Comparisons of Resolution Characteristics

< Unidiagonal Schemes >
a : 2nd order central differences
b : 4th order central differences
c : 6th order central differences
d : Tam's DRP scheme

< Tridiagonal Schemes >
1 : standard Pade' scheme
2 : 6th order tridiagonal scheme (c = 0)
3 : 2nd order optimized tridiagonal scheme
4 : 4th order optimized tridiagonal scheme
5 : 6th order optimized tridiagonal scheme
6 : 8th order tridiagonal scheme

< Pentadiagonal Schemes >
7 : Lele's 4th order spctral-like pentadiagonal scheme
8 : 2nd order optimized pentadiagonal scheme
9 : 4th order optimized pentadiagonal scheme
10 : 6th order optimized pentadiagonal scheme
11 : 8th order optimized pentadiagonal scheme
12 : 10th order pentadiagonal scheme

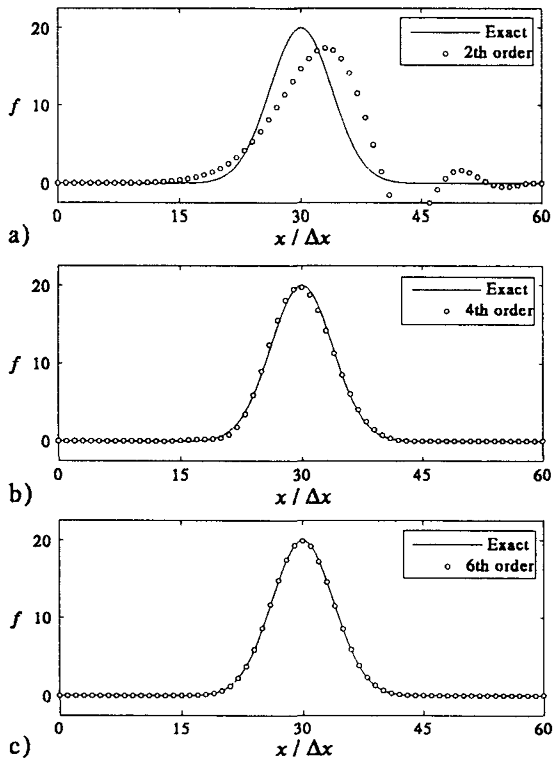

Fig.2 Linear Wave Convection by Tridiagonal Schemes
($CFL = 0.5$, $t = 5000\Delta t$)

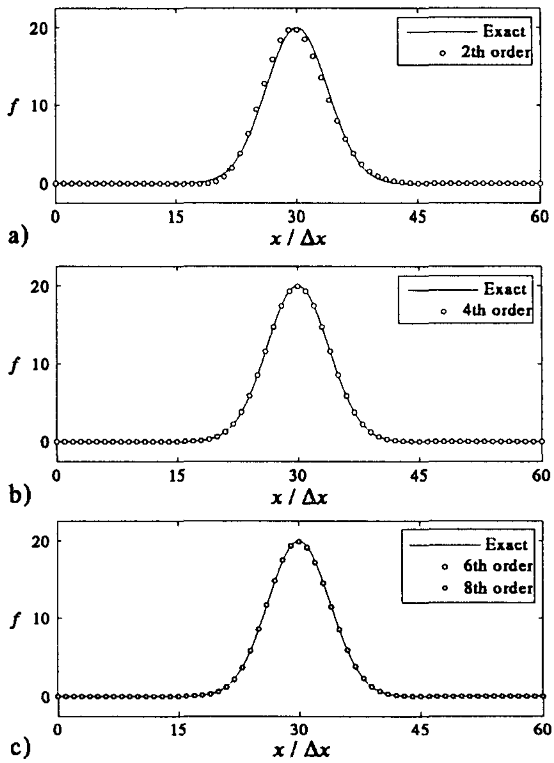

Fig.3 Linear Wave Convection by Pentadiagonal Schemes
($CFL = 0.5$, $t = 5000\Delta t$)

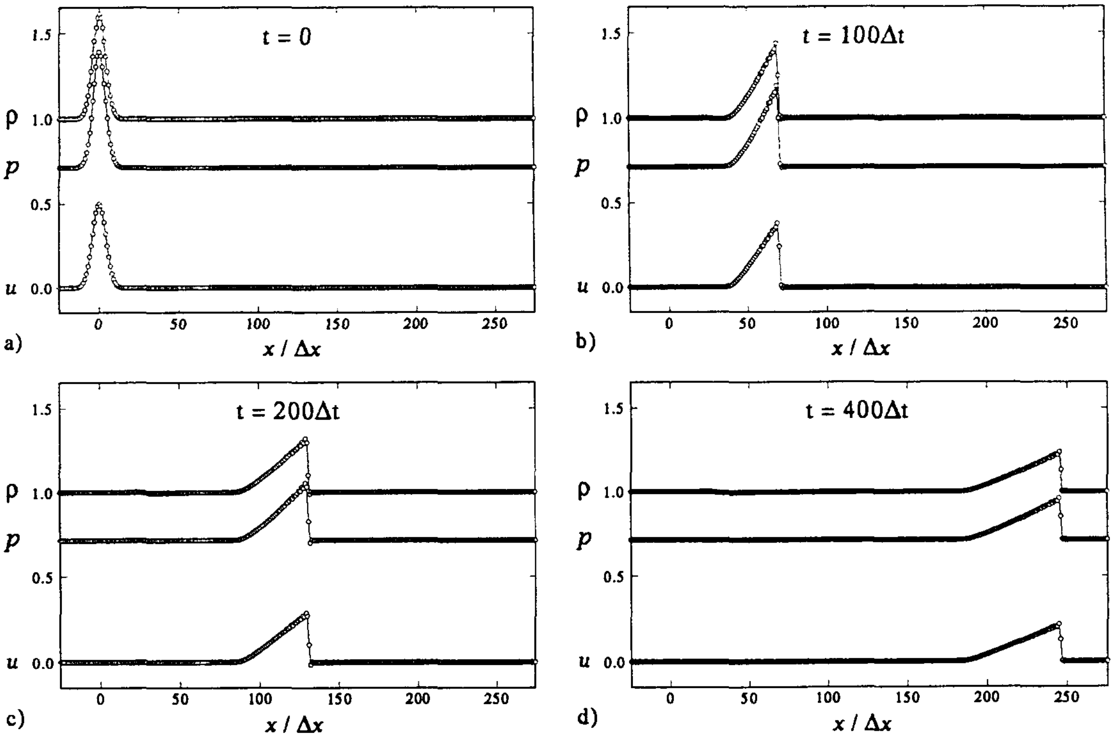

Fig.4 Nonlinear Wave Convection by 6th Order Optimized Tridiagonal Scheme

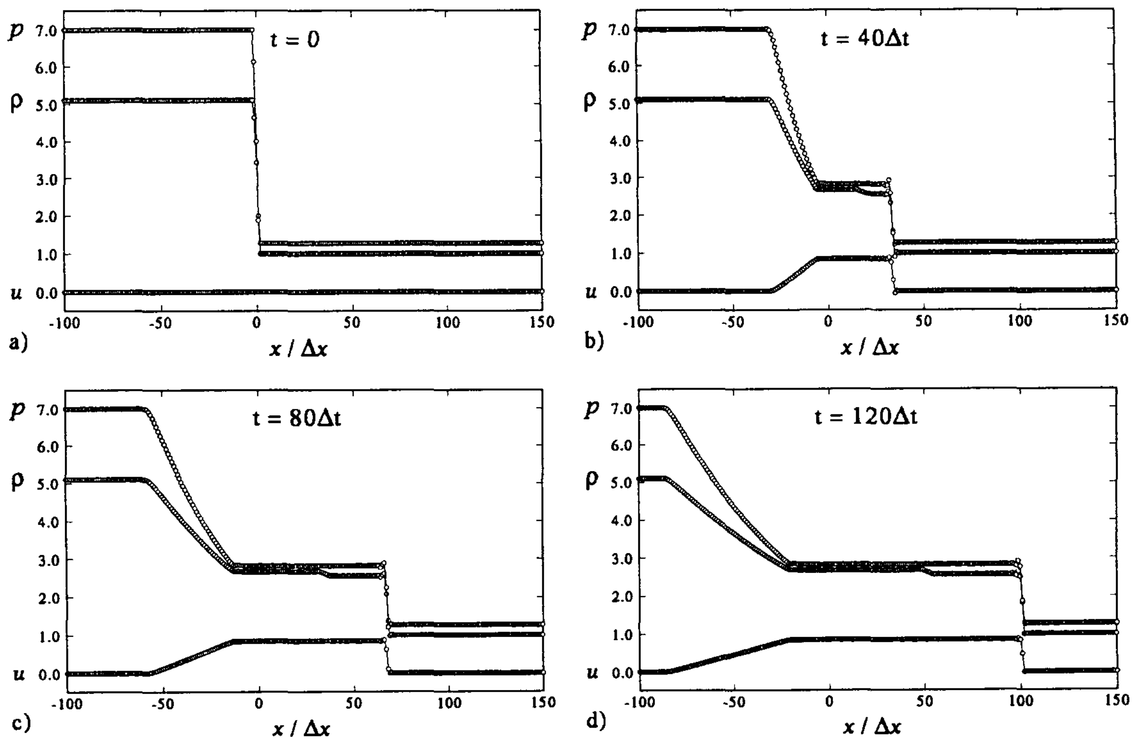

Fig.5 Nonlinear Wave Convection by 6th Order Optimized Tridiagonal Scheme
(Shock Tube Problem)