# Internal Waves of a Two-Layer Fluid with Free Surface over a Semi-circular bump

J. W. Choi
Department of Mathematics, College of Science
Korea University, Seoul, 136-701, Korea

## Abstract

In this paper we study steady capillary-gravity waves in a two-layer fluid bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction. Two critical speeds for the waves are obtained. Near the smaller critical speed, the derivation of the usual forced KdV equation (FKdV) fails when the coefficient of the nonlinear term in the FKdV vanishes. To overcome this difficulty, a new equation called a forced extended KdV equation (FEKdV) governing interfacial wave forms is obtained by a refined asymptotic method. Various solutions and numerical results of this equation are presented.

## 1. Introduction

We consider a two-layer medium of immiscible, inviscid and incompressible fluids having different but constant densities. The medium is bounded above by a free surface and below by a horizontal rigid boundary with an interface in between (Fig. 1). The surface tension effect is taken into consideration at both the free surface and the interface. We assume that a two-dimensional object is moving along the lower boundary at a constant speed, and in reference to a coordinate system moving with the object, the fluid flow is steady. Two critical speeds are obtained. When the object is moving at a speed near either one of them, an FKdV for steady flow can be derived and has been extensively investigated in [1] and [2]. We note that numerical studies of steady flow of a two-layer fluid over a bump or a step bounded by a free or rigid upper boundary were carried out by Forbes [3] and among others, and an asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen [4] on the basis of the FKdV theory, and with surface tension by Choi, Sun and Shen [5], where a forced modified KdV equation (FMKdV) was obtained. The FKdV theory fails when the coefficient of the nonlinear term or that of the third derivative in the FKdV vanishes. In the case considered here, when the wave speed is near the smaller critical speed for internal waves, the amplitude of which is larger at the interface than at the free surface, the coefficient of the nonlinear term in the FKdV may vanish. Furthermore, at a wave speed near either one of the critical speeds the coefficient of the third order derivative may also vanish. To overcome the difficulty of a vanishing nonlinear term in the FKdV, we shall develop a refined asymptotic method to derive a new equation , called the forced extended KdV equation (FEKdV), in the following form:

$$(F_1\eta_2^2 + F_2\eta_2 + F_3)\eta_{2x} + F_4\eta_{2xxx} = -F_5 b_x,$$

where $F_1$ to $F_5$ are constants depending on several parameters and $z = -H^- + b(x)$ is the equation of the obstruction. The objective of this paper is to investigate solutions of the FEKdV, which represent possible interfacial wave forms. We remark that when the coefficient of the third order derivative in the FEKdV vanishes a forced perturbed KdV equation with a fourth order derivative could be derived, and is deferred to a subsequent study. In Section 2, we formulate the problem and develop the asymptotic scheme to derive the FEKdV. Section 3 consists of two subsections. The supercritical case of $F_1F_4 > 0$ and the subcritical case of $F_1F_4 < 0$ are studied in Section 3.1 and 3.2 respectively. In general, we can find three types of solutions. The first-type solution consists of symmetric solitary-wave like solutions. The second-type solution is one which is a part of a free solitary-wave behind the bump and a periodic wave solution ahead of the bump. The free solitary-wave is a solitary- wave solution of the extended KdV equation without forcing. By a third-type solution we mean a solution which is constant behind the bump and periodic ahead of the bump. In many cases both second- and third-type solutions do satisfy the conservation of mass even if they do not tend to zero far upstream. In both sections 3.1 and 3.2, analytical and numerical results, which indicate the appearance of various types of solutions, are presented. It is found that four branches of first-type solutions can appear in the supercritical case and there are no first- and second-type solutions in the subcritical case. The third-type solutions appear in both supercritical and subcritical cases. In both cases symmetric solutions without a periodic part are embeded in the third-type solutions at discrete values of a parameter, and a hydraulic jump wave solution appears as a limting case of third-type solutions in the subcritical case.

## 2. Formulation and Successive Approximate Equations

We consider steady internal capillary-gravity waves between two immiscible, inviscid and incompressible fluids of constant but different densities bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction of compact support. The domains of the upper fluid with a constant density $\rho^{*+}$ and the lower fluid with a constant density $\rho^{*-}$ are denoted by $\Omega^{*+}$ and $\Omega^{*-}$ respectively (Fig. 1). Assume that the small obstruction is moving with a constant speed $C$. In reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed C far upstream. The governing equations and boundary conditions are given by Euler equations.

We define the following nondimensional variables:

$$\epsilon = H/L << 1,\ \ \eta_1 = \epsilon^{-1}\eta_1^*/H^{*-},\ \ \eta_2 = \epsilon^{-1}\eta_2^*/H^{*-},\ \ p^{\pm} = p^{*\pm}/gH^{*-}\rho^{*-},$$
$$(x,z) = (\epsilon x^*, z^*)/H^{*-},\ \ (u^{\pm}, w^{\pm}) = (gH^{*-})^{-1/2}(u^{*\pm}, \epsilon^{-1}w^{*\pm}),$$

$$\rho^+ = \rho^{*+}/\rho^{*-} < 1,\ \rho^- = \rho^{*-}/\rho^{*-} = 1,\ U = C/(gH^{*-})^{1/2}\,,$$
$$T_i = T_i^*/\rho^{*-}gH^{*-},\ i = 1,2,\ h = H^{*+}/H^{*-}, b(x) = b^*(x)(H^{*-}\epsilon^3)^{-1}\,,$$

where $u^{*\pm}$ and $w^{*\pm}$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, $g$ is the gravitational acceleration constant, $T_1^*$ and $T_2^*$ are surface tension constants at the free surface and the interface respectively. $L$ is the horizontal scale, $H$ is the vertical scale, $b(x) = b^*(x)(H^{*-}\epsilon^3)^{-1}$, $H^{*+}$ and $H^{*-}$ are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$ respectively, and $z^* = -H^{*-} + b^*(x)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above equations become in fluid domain,

$$u_x^\pm + w_z^\pm = 0\,, \tag{1}$$
$$u^\pm u_x^\pm + w^\pm u_z^\pm = -p_x^\pm/\rho^\pm\,, \tag{2}$$
$$\epsilon^2 u^\pm w_x^\pm + \epsilon^2 w^\pm w_z^\pm = -p_z^\pm/\rho^\pm - 1\,; \tag{3}$$

at $z = h + \epsilon\eta_1$,

$$p^+ = -\epsilon^3 T_1 \eta_{1xx}/(1 + \epsilon^4\eta_{1x}^2)^{3/2}, \tag{4}$$
$$\epsilon u^+\eta_{1x} - w^+ = 0\,; \tag{5}$$

at $z = \epsilon\eta_2$,

$$\epsilon u^-\eta_{2x} - w^- = 0\,, \tag{6}$$
$$\epsilon u^+\eta_{2x} - w^+ = 0\,, \tag{7}$$
$$p^+ - p^- = \epsilon^3 T_2\eta_{2xx}/(1 + \epsilon^4\eta_{1x}^2)^{3/2}\,; \tag{8}$$

at $z = -1 + \epsilon^3 b(x)$,

$$w^- = \epsilon^3 u b_x\,, \tag{9}$$

where $b(x)$ has a compact support.

In the following, we use a unified asymptotic method to derive the equations for $\eta_1(x)$ and $\eta_2(x)$. We assume that $u^\pm, w^\pm$, and $p^\pm$ are functions of $x, z$ near the equilibrium state $u^\pm = u_0$, $w^\pm = 0$, $p^+ = -\rho^+ z + \rho^+ h$ and $p^- = -\rho^- z + \rho^+ h$, where $u_0$ is a constant, and possess asymptotic expansions:

$$(u^\pm, w^\pm, p^\pm) = (u_0, 0, -\rho^\pm z + \rho^+ h) + \epsilon(u_1^\pm, w_1^\pm, p_1^\pm)$$
$$+ \epsilon^2(u_2^\pm, w_2^\pm, p_2^\pm) + \epsilon^3(u_3^\pm, w_3^\pm, p_3^\pm) + O(\epsilon^4)\,. \tag{10}$$

By inserting (10) into (1) to (9) and arranging the resulting equations according to the powers of $\epsilon$, it follows that $(u_0, 0, -\rho^\pm z + \rho^+ h)$ are the solutions of the zeroth order system of equations. The equations of first to third oder terms of $\epsilon$ are also given and by solving the equations according to the orders we obtain the following time independent extended KdV equation with forcing term:

$$F_1\eta_2^2\eta_{2x} + \lambda_1\eta_2\eta_{2x} + \lambda_2\eta_{2x} + F_4\eta_{2xxx} + F_5 b_x = 0\,, \tag{11}$$

where $F_1$ through $F_4$ are constants. The coefficient $F_1$ contains parameters $T_1$ and $T_2$ and $F_3$ contains a parameter $\lambda$. Since $F_3$ and $F_4$ change signs as $\lambda$ and $T_i$ vary, (11) has different types of solutions for different cases of $T_i$ and $\lambda$.

## 3. Extended KdV Equation with Forcing

The sign of $F_4F_1$ determines the existence of solutions of (11). In the following sections, the two cases $F_4F_1 > 0$ and $F_4F_1 < 0$ will be considered separately. We remark in passing that if the surface tension constants $T_1$ and $T_2$ satisfy $F_4 = 0$ for given $\rho$ and h the coefficient of the third order derivative vanishes and a forced perturbed KdV equation could be derived to replace the FEKdV equation.

### 3.1. Supercritical Case ( $F_4F_1 > 0$ )

We can investigate the behavior of solutions ahead and behind the bump by using elliptic ingetral and show that the solution of (11) always exist over the bump in this case, and a global solution of (11) can be constructed. In the following, we use numerical computation to find various types of solutions of (11) and the equation for the bump is given by $b(x) = (1 - x^2)^{1/2}$ for $-1 \leq x \leq 1$. We devide these solutions into symmetric solitary-wave like solutions, which are first-type solutions, and unsymmetric solutions, which consist of second- and third-type solutions.

(I) Symmetric solitary-wave like solutions

By using shooting method, we can find a symmetric solitary-wave like solution of (11) whose values ahead and behind the bump are given by a solitary-wave solution. The numerical results are presented in Figs.2-3. Four typical solitary-wave like solutions corresponding to $\lambda_1 = -1$ and $\lambda_2 = 4$ are shown in Fig. 2. In Fig. 3, we show the relationship between $\eta_2(0)$ and $\lambda_2$ with $\lambda_1 = -1$. We note that for certain pairs of $(\lambda_1, \lambda_2)$ no solitary-wave like solution can appear.

(II) Unsymmetric solutions

The numerical results are presented in Figs. 4-7. Fig. 4 shows the second-type solutions which have solitary-wave solution as their solutions in $(-\infty, -1]$. In Fig. 5 we show a second-type solution whose mean depth of the wave ahead of the bump is $-\eta_2(-\infty)$. We also consider third-type solutions which are equal to $H_2$ for $x \leq -1$ and periodic for $x \geq 1$. Fig.6 shows a typical third-type solution whose mean depth ahead of bump is $-\eta_2(-\infty)$. An interesting phenomenon appears in the process of computing third-type solutions. At discrete values of $\lambda_2$ there are symmetric solutions without a periodic part embedded in the third-type solutions and Fig. 7 presents such a solution. In Figs. 2-7, we choose $\rho = 0.2$, $h = 0.6$, $\lambda_1 = -1$, $T_1 = 2.8$ and $T_2 = 4$.

### 3.2. Subcritical Case ( $F_4 F_1 < 0$ )

The same assumptions as in 3.1 are given in this section. The only difference here is $F_1 F_4 < 0$. Here we choose $\eta_2 \equiv H_2$ in $(-\infty, x^-)$ since a solitary-wave solution does not exist for (38) when $b(x) = 0$. It can also be shown that (11) possesses a solution with a continous second order derivative in $[x^-, x^+]$ in this case, and by using the matching process as before, we can find the solution for all real x. We present the numerical results of global solutions in Figs. 8-11. Fig. 8 shows a typical third-type solution and Fig. 9 shows a hydraulic jump, which is the limiting solution of the third-type solution as $\lambda_2$ being decreased and tending to some critical value. Only unbounded solutions are found when $\lambda_2$ is decreased further below the critical value. We also find multi-troughs symmetric solutions without a periodic part embeded in the third-type solutions at discrete values of $\lambda_2$, even though no first-type symmetric solutions exist in the subcritical case. Figs. 10-11 present two such cases. In Figs. 8 to 11 we choose the same $\rho$ and h as in Figs. 4 to 7, but $\lambda_1 = 1$ and $T_1$ and $T_2$ are equal to $10^{-2}$. The obstruction is same as before.

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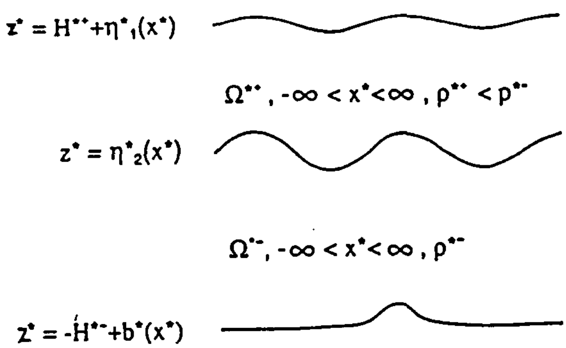


Fig. 1 Fluid Domain

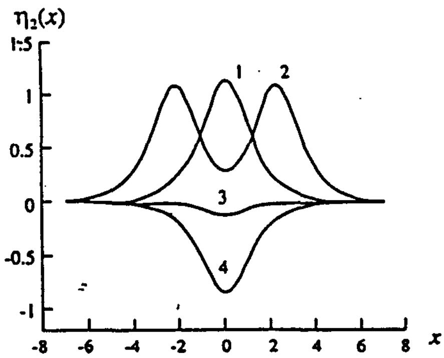


Fig. 2 Four different types of symmetric solutions - Supercritical case. $\lambda = -2$

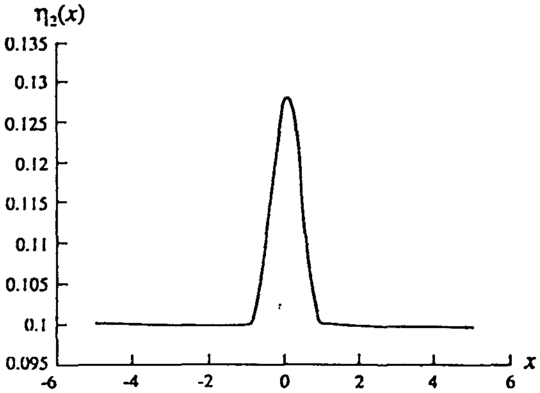


Fig. 7 Symmetric solution with one crest - Supercritical Case

$\lambda_1 = 27.1,\ H_1 = \eta_2(-\infty) = 0.1$

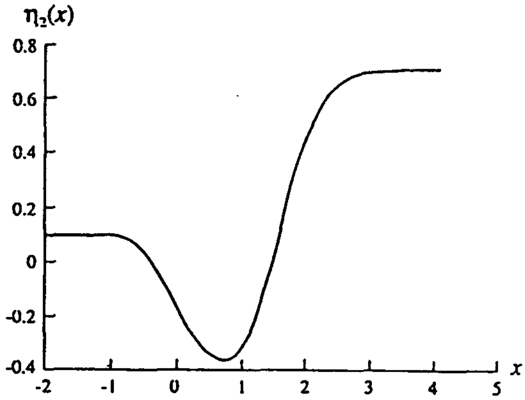


Fig. 9 Hydraulic jump wave - Subcritical Case

$\lambda_1 = 2.031215,\ H_1 = \eta_2(-\infty) = 0.1$

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