댐핑영향을 반영하는 Lyapunov 함수 그룹의 유도 및 응용

문 영 현* 최 병 곤*° 노 태 훈* 이 태 식* 이 윤 섭**
*연세대학교 전기공학과 공주대

Derivation of a Group of Lyapunov Functions reflecting Damping Effects and its Application

Y. H. Moon* B. K. Choi*° T. H. Roh* T. S. Lee* Y. S. Lee**
Dept. of Electrical Eng. Yonsei Univ. Gongju Univ.

Abstract

Most of the theorems of nonlinear stability is based on the Lyapunov stability theory. The Lyapunov function method is the most well-known and provides precise and rigorous theoretical backgrounds. However, the conventional approach to direct stability analysis has been performed without taking account of damping effects. For accurate stability analysis of nonlinear systems, it is required to consider the damping effects.

This paper presents a new method to derive a group of Lyapunov functions to reflect the damping effects by considering the integral relationships of the system governing equations. This method can be utilized as a powerful tool to determine the region of attraction.

Key words : Nonlinear System; Stability Analysis; Direct Method; Lyapunov Function; Energy Function

I. Introduction

In the operation of Nonlinear Systems, the stability problem is the first issue to be solved. Many authors contributed to developing the nonlinear stability theory, yielding the Popov theorem and the ultimate confinement theorem for Lure-Type nonlinear systems.[1]

Most of the theorems of nonlinear stability is based on the Lyapunov stability theory. The Lyapunov function method is the most well-known and provides precise and rigorous theoretical backgrounds. This method has a great merit that it is possible to tell the future system stability by only the present state. However, this method has not been widely used since there is no general method to find appropriate Lyapunov functions. One of the popular methods to find a Lyapunov function is using the energy function of the system. Most of the nonlinear systems have no global Lyapunov function, and thus local Lyapunov functions are generally used to determine a local stability around a certain singular solution in concern. In this case, system damping should be considered as an important factor to determine the accurate local stability. However, there has been no attempt to reflect the damping effects in the Lyapunov function. This paper presents a new method to derive a group of Lyapunov functions to reflect the damping effects by considering the integral relationships of the system governing equations.

II. Local Stability Analysis by using Energy functions

Given a nonlinear system, it is relatively easy to find an energy function E, which satisfy $\frac{dE}{dt} \le 0$ for all time. Assume that we are concerned with the system stability around a certain singular solution x^c. The local stability around x^c can be determined by the following Lemma, which is only a sufficient condition.

<u>Lemma</u> : Assume that there exists a well-defined energy function $E(\mathbf{x})$ for the system concerned. If there is a convex region R_s around x^c and the energy function $E(\mathbf{x})$ is convex ,and if its time derivative along the trajectory of any state in a convex region R_s, is negative semi-definite, then the system is locally stable around x^c, and R_s is a subset of the region of attraction.

The above Lemma can be widely applied to local stability analysis of nonlinear systems if some appropriate energy function is provided. Most of the conventional studies have adopted the dynamic energy function which is given by the sum of kinetic and potential energies, and the conventional all the system losses decrease the system energy with $\frac{dE}{dt} \le 0$. Therefore, the consideration of system losess gives a better estimation of the region of attraction. The Lyapunov condition is just sufficient condition, and there is some possibility to find a wider stable region by considering the system damping.

In this study, it will be shown that a group of local Lyapunov function can be derived to reflect the damping effects by considering the integral relationship of the system governing equations.

III. Derivation of a Group of Lyapunov function by considering system losses

There is no other thumb rule to find a Lyapunov function than investigating case-by-case. The most popular method to find a Lyapunov function is taking the energy function as a Lyapunov function. This study will also show how to derive a group of energy-related Lyapunov functions by considering system losses with the following sample R-L-C system.

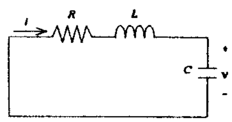


Fig.1 RLC Circuit

The state equation for the above system is given by

$$\dot{v} = \frac{1}{C} i \tag{1.a}$$

$$\dot{i} = -\frac{R}{L} i - \frac{1}{L} v \tag{1.b}$$

The energy function for this system is given by

$$E = \frac{1}{2} L i^2 + \frac{1}{2} C v^2 \tag{2}$$

Then, it can be easily shown that the time derivative energy function is given by

$$\frac{dE}{dt} = -Ri^2 \tag{3}$$

This energy function can be taken as a Lyapunov function. However, the energy function E in Eq.(2) does not reflect the system loss Ri^2. We can derive some other type of Lyapunov function by using integral relations of the system equation.

Equation (1.b) can be rewritten as follows:

$$L\dot{i} = -Ri - v \tag{4}$$

By multiplying $\dot{i}$ to both sides of the above equation, we get

$$\begin{aligned} L\dot{i}^2 &= -Ri\dot{i} - v\dot{i} \\ &= -Ri\dot{i} - vC\ddot{v} \end{aligned} \tag{5}$$

(where $\dot{i} = C\ddot{v}$ from Eq.(1.a) is used)

Integrating both sides of the above equation gives

$$L\int \dot{i}^2 dt = -\frac{1}{2}Ri^2 - C\int v\ddot{v}dt \tag{6}$$

By using Eq. (1.a), integrating the last term of Eq. (6) by part yields

$$\int v\ddot{v}dt = v\dot{v} - \int \dot{v}^2 dt = \frac{1}{C}vi - \frac{1}{C^2}\int i^2 dt \tag{7}$$

By substituting Eq.(7) in Eq.(6), we can obtain

$$\int i^2 dt = \frac{1}{2}RCi^2 + Cvi + LC\int \dot{i}^2 dt \tag{8}$$

For the given system, we have the following energy conservation equation.

$$\frac{1}{2}Li^2 + \frac{1}{2}Cv^2 + \int Ri^2 dt = K \tag{9}$$

where K is a constant number.

The constant number K can be enterpreted as an initial energy of system. The substitution of Eq.(8) into Eq.(9) yields

$$\frac{1}{2}(L + R^2C)i^2 + CRvi + \frac{1}{2}Cv^2 + LCR\int \dot{i}^2 dt = K \tag{10}$$

If the system parameters satisfy the following condition

$$D = (CR)^2 - (L + R^2C)LCR < 0, \tag{11}$$

we can get another Lyapunov function as follows:

$$E_1 = \frac{1}{2}(L + R^2C)i^2 + CRvi + \frac{1}{2}Cv^2 \tag{12}$$

Here, it is noted that inequality (11) is the necessary condition to ensure the convexity of function E_1.
The time-derivative of E_1 can be given by

$$\frac{dE_1}{dt} = -LCR\dot{i}^2 \tag{13}$$

This relation can be easily proven either by the chain rule or by differentiating the transformed energy conservation rule in Eq.(10). When using the method of the latter, it is required to substitute Eq.(12) into (10) before deferentiating.
By examining Eqs.(12) and (13), we can see that a new energy function E_1 can be another type of energy-related Lyapunov function if the system parameters satisfy the condition (11).
On the other hand, we can derive a group of energy functions by splitting the damping terms. Eq.(9) can be rewritten as follows :

$$\frac{1}{2}Li^2 + \frac{1}{2}Cv^2 + \lambda\int Ri^2 dt + (1-\lambda)\int Ri^2 dt = K \tag{14}$$

with $0 \le \lambda \le 1$

By replacing the first damping-related term by Eq.(8) and rearranging it, we can obtain the following equation.

$$\begin{aligned} &\frac{1}{2}(L + \lambda R^2C)i^2 + \lambda CRvi + \frac{1}{2}Cv^2 \\ &+ \int\left[\lambda LCR\dot{i}^2 + (1-\lambda)Ri^2\right]dt = K \end{aligned} \tag{15}$$

The above equation can be called as variants of the energy conservation law in Eq.(9) with all $\lambda \in [0,1]$.

By examining Eq.(15), we can obtain a group of Lyapunov functions as follows :

$$L_\lambda(i,v) = \frac{1}{2}(L + \lambda R^2C)i^2 + \lambda CRvi + \frac{1}{2}Cv^2 \tag{16}$$

with all $\lambda \in [0,1]$ which satisfies

$$\begin{aligned} D &= (CR)^2\lambda^2 - (L + \lambda R^2C)C \\ &= R^2C^2\lambda^2 - R^2C^2\lambda - LC < 0 \end{aligned} \tag{17}$$

However, it is obvious that any $\lambda \in [0,1]$ satisfies inequality (17), which has the following solution range :

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 + L/(R^2C)} < \lambda < \frac{1}{2} + \frac{1}{2}\sqrt{1 + L/(R^2C)} \tag{17'}$$

The time derivative of Lyapunov function L_λ is given by

$$\frac{d}{dt}L_\lambda(i,v) = -\lambda LCR\dot{i}^2 - (1-\lambda)Ri^2 \le 0 \tag{18}$$

Condition (17) guarantees that $L_\lambda(i,v)$ is a convex function, and Eq.(18) shows the negative semi-definiteness of the time derivative of the Lyapunov function.

The above typical example shows that we can derive a group of global Lyapunov function for a simple RLC circuit. This method can be applied to harmonic oscillator problems such as pendulum oscillators and swing equations of power system with constant damping.

IV. Illustrative Example

Consider a one-machine infinite bus system with a pure reactive line as shown in Fig.2. We assume that the system rests at an equilibrium point for all time $t < 0$.

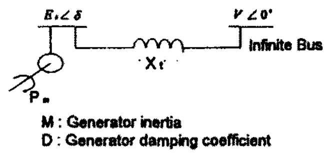


Fig. 2 Single machine infinite bus system

The swing equation of the above system is given by

$$\dot{\delta} = \omega \tag{19.a}$$

$$M\dot{\omega} + D\omega = P_m - \frac{E_t V \sin\delta}{X_t} \tag{19.b}$$

with $\delta(t) = \delta_s$, $\omega(t) = 0$ *for all* $t \le 0$

where M : Generator Inertia
D : Generator Damping Coefficient
P_m : Mechanical Input Power [MW]

For stability analysis of the system, the mechanical input P_m is considered to be constant, and the generator internal voltage E_t is also assumed to be well governed to be constant. Under these assumptions, the conventional system energy is given by

$$E = \frac{1}{2}M\omega^2 + \frac{E_t V(\cos\delta_s - \cos\delta)}{X_t} - P_m(\delta - \delta_s) \tag{20}$$

where δ_s is a singular solution of the swing equations, i.e.

$$\delta_s = \sin^{-1}\left(\frac{X_t P_m}{E_t V}\right) \tag{21}$$

The time derivative of the energy function E is given by

$$\frac{dE}{dt} = -D\omega^2 \tag{22}$$

By integrating Eq.(22) in the time interval [0, t] and equating it with Eq.(20), we can derive the following energy conservation law :

$$\frac{1}{2}M\omega^2 + \frac{E_sV(\cos\delta_s - \cos\delta)}{X_t} - P_m(\delta - \delta_s) + \int_0^t D\omega^2 dt = E_0 \tag{23}$$

with $E_0 = E(0)$: Initial energy of the system

The energy function E has an energy well around the singular solution as shown in Fig. 3.

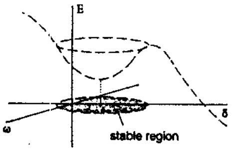


Fig. 3 Energy well of function E

The energy function E directly gives the stable region S shadowed in Fig.3. However, energy function E does not reflect the damping effects. When the damping coefficient is significantly large, the stable region should be extended.

Derivation of a Energy Function to reflect Damping Effects

In order to reflect the damping effects, it is necessary to change the damping-related term in the energy conservation law(23) into an appropriate form including a path independent energy term. This can be done by using integral relationships derived from system governing equations (19.a) and (19.b). Here, we can derive an integral relation from Eq.(19.b). First multiplying the both sides of Eq.(19.b) by $\Delta\delta$ and integrating with respect to time t, we obtain

$$\int_0^t M\dot{\omega}\Delta\delta dt + \int_0^t D\omega\Delta\delta dt = \int_0^t (P_m - P_e)\Delta\delta dt \tag{24.a}$$

where $\Delta\delta = \delta - \delta_s$ (24.b)

In the above equation, the angular velocity ω can be represented as follows:

$$\omega = \frac{d\delta}{dt} = \frac{d\Delta\delta}{dt} \tag{25}$$

Substitution of Eq.(25) in Eq.(24.a) gives

$$\int_0^t M\Delta\ddot{\delta}\Delta\delta dt + \int_0^t D\Delta\dot{\delta}\Delta\delta dt = \int_0^t (P_m - P_e)\Delta\delta dt \tag{26.a}$$

$$\text{where} \quad P_e = \frac{E_sV\sin\delta}{X_t} \tag{26.b}$$

Here, the first term in Eq.(26.a) can be rewritten as follows:

$$\int_0^t M\Delta\ddot{\delta}\Delta\delta dt = M\Delta\dot{\delta}\Delta\delta\Big|_0^t - \int_0^t M\Delta\dot{\delta}^2 dt$$

$$= M\omega\Delta\delta - M\int_0^t \omega^2 dt \tag{27}$$

Since the second term of Eq.(26.a) can be directly integrated, the substitution of Eq.(27) into Eq.(26.a) gives.

$$M\omega\Delta\delta - M\int_0^t \omega^2 dt + \frac{1}{2}D\Delta\delta^2 = \int_0^t (P_m - P_e)\Delta\delta dt$$

From the above equation, we can obtain the following integral relationship:

$$\int_0^t \omega^2 dt = \omega\Delta\delta + \frac{1}{2}\frac{D}{M}\Delta\delta^2 - \frac{1}{M}\int_0^t (P_m - P_e)\Delta\delta dt \tag{28}$$

By using this relationship and splitting the damping loss term in a similar way as given in Eq.(14), we can derive a group of energy functions from Eq.(23) as follows :

$$E_\lambda(\omega,\delta) = \frac{1}{2}M\omega^2 + \frac{E_sV(\cos\delta_s - \cos\delta)}{X_t} + D\lambda\omega\Delta\delta + \frac{1}{2}\frac{D^2}{M}\lambda\Delta\delta^2 - P_m(\delta - \delta_s) \tag{29}$$

with $0 \le \lambda \le 1$

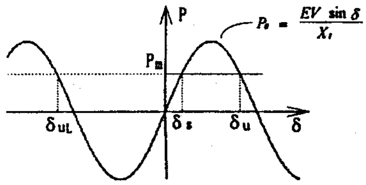
The time-derivative of E_λ is given by

$$\frac{dE_\lambda}{dt} = -(1-\lambda)D\omega^2 + \frac{\lambda}{M}(P_m - P_e)\Delta\delta \le 0 \quad \text{for all time} \tag{30}$$

In the above equation, the last term has seminegativeness since singular solution δ_s can be determined by the following power curve in Fig.4, which shows that

$$(P_m - P_e)\Delta\delta \le 0 \qquad \text{for all time.}$$

As a result, the seminegativeness of the time-derivative of E_λ can be always guaranteed for all $\lambda \in [0,1]$



If $\delta_s < \delta < \delta_u$, *then* $P_m - P_e < 0$

If $\delta_{uL} < \delta < \delta_s$, *then* $P_m - P_e > 0$

Fig.4 Power Curve

Now, we will examine the local convexness of the energy function E_λ around the singular solution. E_λ can be approximated around $\delta = \delta_s$ as follows:

$$E_\lambda \cong \frac{1}{2}M\omega^2 + \frac{E_sV\sin\delta_s}{X_t}\Delta\delta + \frac{1}{2}\frac{E_sV\cos\delta_s}{X_t}\Delta\delta^2 + D\lambda\omega\Delta\delta + \frac{1}{2}\frac{D^2}{M}\lambda\Delta\delta^2 - P_m\Delta\delta$$
$$= \frac{1}{2}M\omega^2 + D\lambda\omega\Delta\delta + \frac{1}{2}\left(\frac{E_sV\cos\delta_s}{X_t} + \frac{D^2}{M}\lambda\right)\Delta\delta^2 \tag{31}$$

In the above equation, the following relationship is used,

$$P_m = \frac{E_sV\sin\delta_s}{X_t} \tag{32}$$

In Eq. (31), E_λ can be convex around X_s if

$$(D\lambda)^2 - M\left(\frac{E_sV\cos\delta_s}{X_t} + \frac{D^2}{M}\lambda\right) < 0 \tag{33}$$

This inequality has the following solution range :

$$\frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 + \frac{ME_s\cos\delta_s}{D^2X_t}} < \lambda < \frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \frac{ME_s\cos\delta_s}{D^2X_t}} \tag{34}$$

Since $|\delta_s|$ should be less than $\pi/2$, any $\lambda \in [0,1]$ always satisfies inequality (34). As a result, it can be concluded that, for any $\lambda \in [0,1]$, energy function E_λ can be used as a local Lyapunov function.

Determination of the region of Attraction

By selecting an appropriate λ, we can establish a local Lyapunov function E_λ, and can easily find a stable region S_λ associated with E_λ. Then, an estimate of the region of attraction for the system is given by

$$S = \bigcup_{\lambda \in [0,1]} S_\lambda \tag{35}$$

With the changes of λ, the stable region S_λ also continuously changes. Fig.5 shows the variation of S_λ with the changes of λ. The region of attraction can be determined by the shadowed area, and the dark area is the stable region determined by the conventional energy function with the no account of the damping effect. This illustrates that the proposed method can be an effective means to take into account the damping effect in stability analysis. Here it is noted that the region S is a subset of the actual region of attraction. There is high possibility that the system may have some attraction region outside of the region $\delta_{uL} < \delta < \delta_u$, where δ_{uL} is an another equilibrium point located in the left side of the stable equilibrium point. However, we have almost no concern with those stable region since the system is always operated in the region $\delta_{uL} < \delta < \delta_u$, which is the center part in Fig.5.

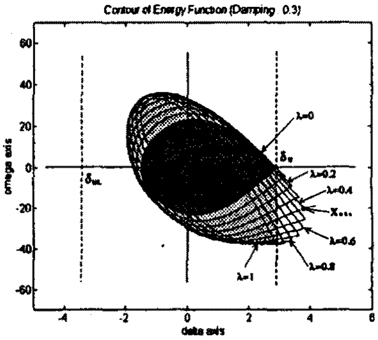


Fig. 5 Stable region S_λ with the changes of λ

Numerical Results

Numerical analysis has been carried out for the system in Fig. 2 with the following parameters :

$H = 4.0$ rated frequency $f_0 = 60Hz$
$X_t = 0.28$ $P_m = 1.00$ pu
$E_s = 1.211$ $M = H/\pi f_0 = 0.0212$

The damping coefficient D can be varied with the generator design, mainly depending on the damping winding. Most of generators have the damping of 1-3 [pu]. Some generators may have a much high value up to 25 [pu] and some may have a quite less value.[4] In this study, we have analyzed the system stability by using various damping coefficients ranging from 0.3 to 3.0 [pu].

i) In case of D=0.3 [pu]

In this case, we have observed the behaviour of the damping-reflected energy function E_λ with the parameter λ changing from 0 to 1 by the step of 0.1. The saddle point of E_λ, say $X_{u\lambda}$, can be calculated from the following equations :

$$\frac{\partial E_\lambda}{\partial \omega} = M\omega + D\lambda\Delta\delta = 0$$

$$\frac{\partial E_\lambda}{\partial \delta} = \frac{EV\sin\delta}{X_t} + D\lambda\omega + \frac{D^2}{M}\lambda\Delta\delta - P_m = 0$$

The equipotentials of $E_\lambda(X_{u\lambda})$ is shown in Fig. 5(given above). In this case, the stable region S_λ determined by E_λ varies without much expansion of the stable region as parameter λ increases. The region of attraction should be determined by taking the union of S_λ's for all $\lambda \in [0,1]$ as mentioned before. The shaded region in the upper half plane is indeed a stable region, where all states move right and cannot pass away any equipotential line. Therefore, all states must eventually be captured by the dark region which is an ultimate confinement region. The shaded region in the lower half plane is also stable region since all states in the region $\omega < 0$ moves left and cannot pass away any equipotential line. Therefore, all states in the shaded region of the lower half plane enter into the dark region or the upper shaded region, which guarantees the stability of the region. when the state arrives $\delta = \delta_u$ with $\omega > 0$ it must be captured by dark region.

ii) In case of D=0.5 [pu]

In this case, we also tried to apply the same procedure as the above case. However, we have got some troubles to determine stable region. The equipotentials of $E_\lambda(X_{u\lambda})$ are obtained with the manner used in the former case and they are shown with the dot lines in Fig. 6. The dot line for $\lambda = 0.2$ includes the region where $\delta > \delta_u$ and $\omega > 0$, in which the seminegativeness of dE/dt is not guaranteed and the system state moves to right side away from the stable region. The stable region should be determined so that there is no possibility that the system state may outer such a region. In order to achieve this purpose, this study adopts the equipotential of $E_\lambda(\delta_u, 0)$ than $E_\lambda(X_{u\lambda})$, when the unstable equilibrium point $(\delta_u, 0)$ is encircled by the equipotential of $E_\lambda(X_{u\lambda})$. The equipotentials of $E_\lambda(\delta_u, 0)$ are shown with solid lines with λ changing from 0 to 1.0 by the step of 0.1. The shaded region in Fig. 6 is a stable region in the same reason as mentioned in the former case. Here, it is noted that the range of ω in Fig. 6 is restricted to $|\omega| < 100$ rad/sec, since the system state seldom moves beyond this range even in the abnormal state. It is also interesting to consider the stability of the striped region. In this region, all states must move to the left but may pass away the equipotential lines since dE/dt may be positive. However, we conjecture that this region would be a stable region with high possibility. The proof is remained for the further study.

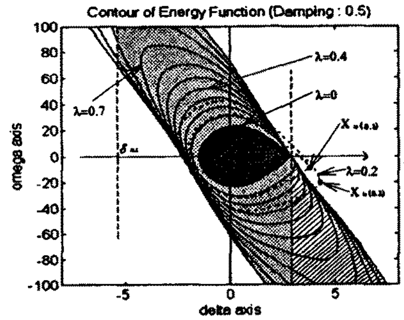


Fig. 6 Stable Region in Case of D=0.5

V. CONCLUSIONS

This paper presents a new method to derive a group of energy-related Lyapunov functions to reflect the damp effects by considering the system governing equations. For stability analysis of nonlinear systems, the local stability has been discussed in order to reflect damping effects. A systematical approach has been developed to convert some part of the damping loss into some appropriate system energy terms. An illustrative example has shown that the proposed method can be well applied to harmonic oscillator problems. By using the Lyapunov function reflecting damping effects, a precise method is presented to determine the region of attraction. The proposed method remarkably improves the accuracy of stability analysis by the direct method.

References

1. Aggarwal, J.K., Notes on Nonlinear Systems, published by D. Van Nostrand Co., 1972.
2. Fouad, A. A. and V. Vittal, Power System Transient Stability Analysis using the transient Energy Function Method, 1989.
3. Moon, Y. H., et al., Integrated Stability Analysis in Power Systems, KOSEF Report, No. 91-0100-03, 1994.