Annual Meeting of Korean Society of Coastal and Ocean Engineers
Sung Kyun Kwan University, 6-7 October 1995

TIME-DEPENDENT WAVE EQUATIONS ON BOTTOM WITH SUBSTANTIAL DEPTH VARIATION

Kyung Duck Suh[1] and Changhoon Lee[2]

1 Introduction

The model for the combined refraction and diffraction of water waves, the so-called mild-slope equation, was first developed by Berkhoff (1972) and has been studied by many coastal engineers because the model is able to consider the combined effect of refraction and diffraction of water waves and eliminate the problem of ray crossing which may happen in the previously developed ray theory.

The time-dependent mild-slope equation was first developed by Smith and Sprinks (1975) by using the Green's formula and, as an equivalent model, a canonical form of the time-dependent mild-slope equations was developed based on the Hamiltonian theory of water waves by Radder and Dingemans (1985). On the other hand, Kubo et al. (1992) developed another type of time-dependent mild-slope equation by expanding the terms in the Berkhoff's equation into Taylor series in $\Delta\omega$ and substituting a term with $\Delta\omega$ by a time-dependent term. The linear dispersive properties of the time-dependent equations were verified by Kirby et al. (1992) and Kubo et al. (1992), who showed the propagation of wave groups using the Radder and Dingemans' model and Kubo et al.'s model, respectively.

In the aforementioned models, the mild-slope assumption $\nabla h/kh << 1$ (where ∇ = horizontal gradient operator, h = water depth, and k = wave number) was made so that the terms of $O(\nabla h)^2$ and $O(\nabla^2 h)$ were neglected. Recently, by using the Galerkin-eigenfunction method, Massel (1993) developed an equation which includes these higher-order bottom effect terms as well as the evanescent modes. In the present study, by using the Green's formula method and the Lagrangian formulation, we develop two equivalent time-dependent wave equations for the propagating wave mode on bottom with substantial depth variation. The developed equation is then applied for the case of a monochromatic wave propagating over a plane slope to show its capability for substantial depth variation.

2 Derivation of the wave equations

2.1 Green's formula method

In the classical linear theory of water waves, the velocity potential $\phi(x, y, z, t)$ for waves with angular frequency ω is governed by

$$\nabla^2\phi + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{1}$$

$$\frac{\partial\phi}{\partial z} = -\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} \qquad z = 0 \tag{2}$$

$$\frac{\partial\phi}{\partial z} = -\nabla h \cdot \nabla\phi \quad z = -h \tag{3}$$

[1]Coastal Engineering Division, Korea Ocean Research & Development Institute
[2]Department of Civil Engineering, Myong Ji University

with the dispersion relationship $\omega^2 = gk \tanh kh$. From the fact that the constant depth solution is the leading order solution for slowly varying depth, the velocity potential ϕ for the propagating waves can be described as

$$\phi(x,y,z,t) = f(x,y,z)\tilde{\phi}(x,y,t) = \frac{\cosh k(h+z)}{\cosh kh}\tilde{\phi} \tag{4}$$

where $f(x,y,z)$ is determined by the local water depth and wavenumber. Applying the Green's second identity to f and ϕ as

$$\int_{-h}^{0}\left(f\frac{\partial^2\phi}{\partial z^2} - \phi\frac{\partial^2 f}{\partial z^2}\right)dz = \left[f\frac{\partial\phi}{\partial z} - \phi\frac{\partial f}{\partial z}\right]_{-h}^{0} \tag{5}$$

and using (1) - (4) gives

$$\frac{\partial^2\tilde{\phi}}{\partial t^2} - \nabla\cdot(CC_g\nabla\tilde{\phi}) + (\omega^2 - k^2CC_g)\tilde{\phi} - g\left(f\nabla f\cdot\nabla h|_{-h} + \int_{-h}^{0} f\nabla^2 f dz\right)\tilde{\phi} = 0 \tag{6}$$

where the terms in the last parenthesis are of order $O(\nabla h)^2$ and $O(\nabla^2 h)$ and were neglected in the equation of Smith and Sprinks (1975). Here, we keep these terms to include the effect of substantial bottom variation. Using the following relationships

$$\nabla f = \left(\frac{\partial f}{\partial h} + \frac{\partial f}{\partial k}\frac{\partial k}{\partial h}\right)\nabla h \tag{7}$$

$$\nabla^2 f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial k}\frac{\partial k}{\partial h}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial h}\frac{\partial h}{\partial y} + \frac{\partial f}{\partial k}\frac{\partial k}{\partial h}\frac{\partial h}{\partial y}\right) \tag{8}$$

after a lengthy algebraic manipulation, we obtain

$$\frac{\partial^2\tilde{\phi}}{\partial t^2} - \nabla\cdot(CC_g\nabla\tilde{\phi}) + (\omega^2 - k^2CC_g)\tilde{\phi} + \omega^2\{R_1(\nabla h)^2 + R_2\nabla^2 h\}\tilde{\phi} = 0 \tag{9}$$

where

$$R_1 = \frac{1}{\cosh^2 kh}(W_1I_1 + W_2I_2 + W_3I_3 + W_4I_4 + W_5I_5 + W_6) \tag{10}$$

$$R_2 = \frac{1}{\cosh^2 kh}(U_1I_1 + U_2I_2 + U_3I_3) \tag{11}$$

The expressions of W_i, U_i, and I_i are given in the appendix. For a monochromatic wave, the model equation becomes

$$\nabla\cdot(CC_g\nabla\tilde{\phi}) + k^2CC_g\tilde{\phi} - \omega^2\{R_1(\nabla h)^2 + R_2\nabla^2 h\}\tilde{\phi} = 0 \tag{12}$$

which is the same as the equation developed by Massel (1993) except minor algebraic errors in the Massel's equation.

2.2 Lagrangian formulation

The variational principle governing irrotational fluid motion is given by (Luke, 1967)

$$\delta\int_t\int_x\int_y L\left(x,y,t,\phi,\nabla\phi,\frac{\partial\phi}{\partial t},\eta\right)dydxdt = 0 \tag{13}$$

or the integral of the Lagrangian L over all space and time is stationary with respect to small variation. Luke (1967) gives

$$L = \int_{-h}^{\eta} \frac{p}{\rho} dz = -\frac{1}{\rho} \int_{-h}^{\eta} \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial z} \right)^2 + gz \right] dz \tag{14}$$

Retaining the terms in second order in the Lagrangian L, we get

$$-\rho L = \eta \frac{\partial \tilde{\phi}}{\partial t} + \frac{1}{2} \frac{CC_g}{g} (\nabla \tilde{\phi})^2 + \frac{1}{2} \frac{\omega^2 - k^2 CC_g}{g} \tilde{\phi}^2 + \frac{1}{2} \int_{-h}^{0} (\nabla f)^2 dz \tilde{\phi}^2 + \int_{-h}^{0} f \nabla f dz \tilde{\phi} \nabla \tilde{\phi} + \frac{1}{2} g \eta^2 \tag{15}$$

The stationariness of L with respect to η gives

$$\frac{\partial \tilde{\phi}}{\partial t} = -g\eta \tag{16}$$

which is the linearized dynamic free surface boundary condition. The stationariness of L with respect to $\tilde{\phi}$, after integrating by parts the fifth term on the right side of (15) and using the Leibnitz rule, gives

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \left(\frac{CC_g}{g} \nabla \tilde{\phi} \right) + \frac{\omega^2 - k^2 CC_g}{g} \tilde{\phi} - \left(f \nabla f \cdot \nabla h |_{-h} + \int_{-h}^{0} f \nabla^2 f dz \right) \tilde{\phi} \tag{17}$$

The last term in the preceding equation is equivalent to the last term in (6). Therefore, the preceding equation can be written as

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \left(\frac{CC_g}{g} \nabla \tilde{\phi} \right) + \frac{\omega^2 - k^2 CC_g}{g} \tilde{\phi} + \frac{\omega^2}{g} \{ R_1 (\nabla h)^2 + R_2 \nabla^2 h \} \tilde{\phi} \tag{18}$$

Equations (16) and (18) constitute a canonical form of time-dependent wave equations on bottom with substantial depth variation. The terms in the last brace of the preceding equation are of order $O(\nabla h)^2$ and $O(\nabla^2 h)$ and were neglected in the equation of Radder and Dingemans (1985). The surface displacement η may be eliminated from (16) and (18) in order to obtain (9), proving the equivalence of the two models which are developed in different methods.

3 Conservation of wave action

The validity of the present equation may be demonstrated by showing that it correctly leads to the conservation equation of wave action. The function $\tilde{\phi}$ may be defined as

$$\frac{i}{g} \tilde{\phi} = R e^{i\psi}, \quad R = \tfrac{a}{\omega} \tag{19}$$

where R and ψ are taken to be purely real quantities. The phase ψ has the following relations with wavenumber $\vec{k}$ and angular frequency ω:

$$\vec{k} = \nabla \psi, \quad \omega = -\tfrac{\partial \psi}{\partial t} \tag{20}$$

After substituting (19) into (9), the properties of R and ψ may be used to set the real and imaginary parts of the resulting expression equal to zero. The imaginary part of (9) is given by

$$2\omega \frac{\partial R}{\partial t} + \frac{\partial \omega}{\partial t} R + \vec{k} \cdot (CC_g \nabla R) + \nabla \cdot (\vec{k} CC_g R) = 0 \tag{21}$$

After multiplying by R and using the fact that

$$\vec{k}CC_g = (kC)\left(C_g\frac{\vec{k}}{k}\right) = \omega\vec{C}_g \tag{22}$$

(21) becomes

$$\frac{\partial}{\partial t}(\omega R^2) + \nabla\cdot(\vec{C}_g\omega R^2) = 0 \tag{23}$$

which can be reduced to the conservation equation of wave action:

$$\frac{\partial A}{\partial t} + \nabla\cdot(\vec{C}_g A) = 0 \tag{24}$$

where the wave action A is defined as

$$A = \frac{1}{2}\rho g\frac{a^2}{\omega} \tag{25}$$

This equation has been shown to be the correct conservation relation for an energylike quantity by numerous investigators, including Whitham (1967) and Bretherton and Garrett (1968).

4 A simple test of the model equation

By numerical computation for the reflection coefficient of a monochromatic wave propagating over a plane slope both ends of which are connected to a constant-depth region, Booij (1983) has shown that the mild-slope equation gives accurate results up to 1:3 slope through comparison with a three-dimensional numerical model. Figure 1 shows the comparison among the present equation, Massel's equation, Berkhoff's mild-slope equation and the three-dimensional model results. It is shown that the present model gives reflection coefficients very close to those of the three-dimensional model and the reflection coefficient becomes stable even for very steep slope, while the mild-slope equation underpredicts the reflection coefficient for steeper slopes. The Massel's equation gives larger reflection coefficients for very steep slopes probably due to the errors in his equation. Without the errors, the results should be the same as the present model results.

5 Conclusion

Two time-dependent wave equations to include the effect of substantial depth variation have been developed by different theoretical approaches, but the resulting equations are shown to be identical. For a monochromatic wave, the developed equation reduces to the Massel's (1993) equation for propagating wave alone. The validity of the present equation has been demonstrated by showing that it correctly leads to the conservation equation for wave action without further approximation. The capability of the present equation for substantial depth variation has partially been verified by applying it to waves propagating over a plane slope with different inclination. The developed equation is of time-dependent form so that it can be used for the propagation of random waves though it has been tested only for a monochromatic wave in the present study. The practical engineering application of the equation may include waves propagating over a trench, Bragg reflection from bottom undulation, and so on.

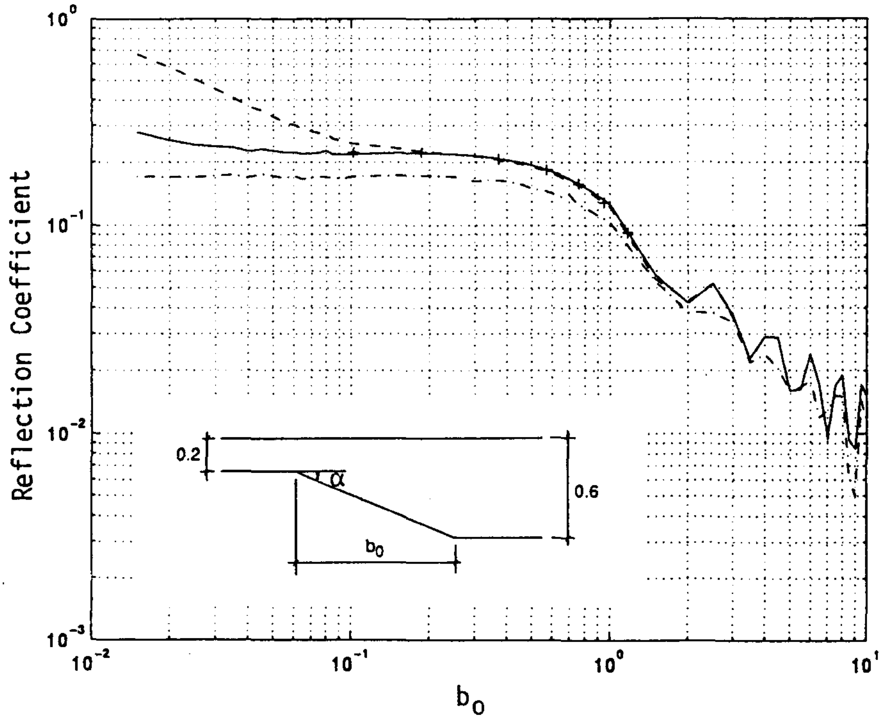

Figure 1: Reflection coefficient versus bottom inclination of a plane slope; + = three-dimensional model, — · — · — = mild-slope equation, ——— = present equation, - - - - - = Massel's equation.

6 Appendix: Components of terms R_1 and R_2

$$W_1 = -2\lambda + 2\frac{f^{(1)}}{k} + 2\frac{khf^{(1)}}{\lambda} - 4\frac{\lambda hf^{(1)}}{k} + \frac{hf^{(2)}}{k} + \frac{h^2f^{(1)2}}{\lambda} - 2\frac{\lambda h^2 f^{(1)2}}{k^2} \quad (26)$$

$$W_2 = 2k - 2\frac{f^{(1)}}{\lambda} + 2hf^{(1)} \quad (27)$$

$$W_3 = 2f^{(1)} - \frac{f^{(2)}}{\lambda} + 2\frac{hf^{(1)2}}{k} \quad (28)$$

$$W_4 = -2\frac{kf^{(1)}}{\lambda} \quad (29)$$

$$W_5 = -\frac{f^{(1)2}}{\lambda} \quad (30)$$

$$W_6 = 1 + \frac{hf^{(1)}}{k} \quad (31)$$

$$U_1 = 1 + \frac{hf^{(1)}}{k} \quad (32)$$

$$U_2 = -\frac{k}{\lambda} \quad (33)$$

$$U_3 = -\frac{f^{(1)}}{\lambda} \quad (34)$$

$$I_1 = \int_{-h}^{0} \cosh^2 k(h+z)dz \tag{35}$$

$$I_2 = \int_{-h}^{0} \cosh k(h+z) \sinh k(h+z)dz \tag{36}$$

$$I_3 = \int_{-h}^{0} (h+z) \cosh k(h+z) \sinh k(h+z)dz \tag{37}$$

$$I_4 = \int_{-h}^{0} (h+z) \cosh^2 k(h+z)dz \tag{38}$$

$$I_5 = \int_{-h}^{0} (h+z)^2 \cosh^2 k(h+z)dz \tag{39}$$

$$f^{(1)} = \frac{\partial k}{\partial h} = -\frac{\lambda k}{\lambda h + \sinh^2 kh} \tag{40}$$

$$f^{(2)} = \frac{\partial^2 k}{\partial h^2} = -\frac{\lambda f^{(1)}}{\lambda h + \sinh^2 kh}\left(2 + \frac{k + hf^{(1)}}{\lambda} \sinh 2kh\right) \tag{41}$$

$$\lambda = \frac{\omega^2}{g} \tag{42}$$

7 References

Berkhoff, J. C. W. (1972). "Computation of combined refraction-diffraction." *Proc. 13th Coast. Engrg. Conf.*, ASCE, Vancouver, 471-490.

Booij, N. (1981). "Gravity waves on water with non-uniform depth and current." Ph.D. dissertation, Delft University of Technology.

Booij, N. (1983). "A note on the accuracy of the mild-slope equation," *Coast. Engrg.*, 7, 191-203.

Bretherton, F. P. and Garrett, C. J. R. (1968). "Wavetrains in inhomogeneous moving media," *Proc. Roy. Soc. London, Series A*, 302, 529-554.

Kirby, J. T., Lee, C., and Rasmussen, C. (1992). "Time-dependent solutions of the mild-slope wave equation," *Proc. 23rd Coast. Engrg. Conf.*, ASCE, Venice, 391-404.

Kubo, Y., Kotake, Y., Isobe, M., and Watanabe, A. (1992). "Time-dependent mild slope equation for random waves," *Proc. 23rd Coast. Engrg. Conf.*, ASCE, Venice, 419-431.

Luke, J. C. (1967). "A variational principle for a fluid with a free surface," *J. Fluid Mech.*, 27, 395-397.

Massel, S. R. (1993). "Extended refraction-diffraction equation for surface waves," *Coast. Engrg.*, 19, 97-126.

Radder, A. C. and Dingemans, M. W. (1985). "Canonical equations for almost periodic, weakly nonlinear gravity waves." *Wave Motion*, 7, 473-485.

Smith, R. and Sprinks, T. (1975). "Scattering of surface waves by a conical island." *J. Fluid Mech.*, 72, 373-384.

Whitham, G. B. (1967). "Non-linear dispersion of water waves," *J. Fluid Mech.*, 27, 399-412.