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A CLOSED-FORM SOLUTION FOR TURBULENT WAVE BOUNDARY LAYERS

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INTRODUCTION

The oscillatory boundary layer that develops when surface waves propagate over the sea bottom affects many flow-dependent phenomena in the coastal zone. Examples of such phenomena are wave energy dissipation due to bottom friction and the initiation and transport of sediment (Grant and Madsen 1986). In nature the boundary layer under waves will almost always be turbulent (Nielsen 1992). Most approaches to calculate the velocity in the wave boundary layer employ the eddy viscosity concept to model the turbulence. The most simple models are based on a linear variation in the eddy viscosity ν_e with elevation z (Grant and Madsen 1979), whereas more complex models assume several different layers, each having a separate equation to relate ν_e and z (Kajiura 1968, Brevik 1981, Myrhaug 1982). In reality, ν_e should also depend on time and Trowbridge and Madsen (1984) developed a model where a time-varying eddy viscosity was employed. However, in most models the prediction of the velocity in the wave boundary layer is not overly sensitive to the formulation of ν_e, and a simple model such as the one suggested by Grant and Madsen (1979) often yield satisfactory results.

The main objective of the present study is to develop a simple, analytical model of the flow in an oscillatory boundary layer under rough turbulent conditions that may be employed for situations where the free-stream velocity is not purely sinusoidal. It will be assumed that the effects of the nonlinear terms in the momentum equations are small enough to be neglected, implying that the linearized boundary layer equation may be used. A simple eddy viscosity formulation in accordance with Grant and Madsen (1979) is employed to model the turbulent stresses. The model is tested with data from Jonsson (1980) for a sinusoidal free-stream velocity and with data from Nadaoka et al. (1994) for an asymmetrical free-stream velocity of cnoidal type.

THEORETICAL CONSIDERATIONS

Employing the simple eddy viscosity model by Grant and Madsen (1979), the linearized turbulent boundary layer (TBL) equation may be written (Nielsen 1992),

$$\frac{\partial}{\partial t}(u_w - u_b) = \frac{\partial}{\partial z}\left\{\kappa u_{*m} z \frac{\partial}{\partial z}(u_w - u_b)\right\} \tag{1}$$

where $u_w(z,t)$ is the velocity in the TBL, $u_b(t)$ the free-stream (wave) velocity, t time, z a vertical coordinate, κ von Karman's constant ($=0.40$), and u_{*m} a constant, representative bottom shear velocity. With the boundary conditions $u_w=0$ for $z=z_o$, where z_o is the characteristic height of the bottom roughness, and $u_w=u_b$ for $z \to \infty$, Equation 1 has the following general solution,

$$u_w = \int_0^t \frac{\partial}{\partial t}\big(u_b(t-\xi)\big) I_u(\xi,z)\, d\xi + u_{bo} I_u(t,z) \tag{2}$$

where u_{bo} denotes u_b at $t=0$ and,

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$$I_u(t,z) = \frac{2}{\pi}\int_o^\infty e^{-\frac{1}{4}y^2\kappa u_{*m}t}\,\frac{J_o(y\sqrt{z_o})Y_o(y\sqrt{z})-J_o(y\sqrt{z})Y_o(y\sqrt{z_o})}{J_o^2(y\sqrt{z_o})+Y_o^2(y\sqrt{z_o})}\,\frac{dy}{y} \tag{3}$$

in which J_o and Y_o are zero-order Bessel functions of the first and second kind, respectively, and y is a dummy integration variable. Equation 3 corresponds to the solution for a time-independent free-stream velocity employed at $t=0$ ($u_b(t)=u_{bo}$=constant); thus, the solution for any $u_b(t)$ is obtained through the superposition of the response from an infinite number of temporal changes in u_b, as expressed by the convolution integral in Equation 2. Figure 1 illustrates I_u in non-dimensional form, and the curves may be interpreted as velocity profiles at different times indicating the TBL growth through the water column.

Equation 2 is a general solution to Equation 1 for any type of free-stream velocity, although from a physical point of view the solution only makes sense for a wave boundary layer where the assumptions behind Equation 1 are applicable. There is only one free parameter in the solution, namely the roughness length scale z_o, which in the case of rough turbulent flow over a flat bed is typically set to $k_n/30$, where k_n is the equivalent Nikuradse sand grain roughness (Grant and Madsen 1979). The representative shear velocity u_{*m} is obtained implicitly from the solution. Grant and Madsen (1979) studied the TBL under a sinusoidal free-stream velocity and used the maximum bottom shear stress τ_{bmax} during a wave period to define $u_{*m}=(\tau_{bmax}/\rho)^{1/2}$, where ρ is the water density. For more complex variations in the free-stream velocity other choices to define u_{*m} may be more appropriate, such as the mean absolute value of the bottom shear stress during a cycle (τ_{bav}).

Equation 2 was derived using Laplace transform technique and for elementary $u_b(t)$ more convenient forms than Equation 2 may be obtained. If the free-stream velocity is sinusoidal and described by $u_b=u_o\cos\omega t$, where u_o is the velocity amplitude at the bottom and ω is the angular frequency, the following solution satisfies Equation 1 and the boundary conditions,

$$\begin{aligned} u_w = {} & u_o\cos\omega t - u_o\frac{N_o\left(2\sqrt{f\frac{z}{z_o}}\right)}{N_o(2\sqrt{f})}\cos\left(\omega t+\Phi_o\left(2\sqrt{f\frac{z}{z_o}}\right)-\Phi_o(2\sqrt{f})\right) \\ & -\frac{2u_o}{\pi}\int_o^\infty e^{-\frac{1}{4}\frac{\omega t}{f}y^2}\frac{y^3}{y^4+(4f)^2}\,\frac{J_o\left(y\sqrt{\frac{z}{z_o}}\right)Y_o(y)-Y_o\left(y\sqrt{\frac{z}{z_o}}\right)J_o(y)}{J_o^2(y)+Y_o^2(y)}\,dy \end{aligned} \tag{4}$$

where N_o and Φ_o is the modulus and phase, respectively, of the zero-order Kelvin function $ker_o x$ + $i\, kei_o x$, and $f=\omega z_o/\kappa u_{*m}$. The second term in Equation 4 is a transient that is dampened out quickly for small values on f; in most cases this term is negligible already after a wave period.

Stream function theory (Dean 1965) is convenient for describing nonlinear wave properties, because the theory is valid from deep water up to wave breaking. The bottom orbital velocity at a point under a wave described by stream function theory may be expressed as,

$$u_b = -\frac{2\pi}{L}\sum_{n=1}^{N} nX(n)\cos(n\omega t) \tag{5}$$

where L is the wavelength, $X(n)$ stream function coefficients, and N the order of the theory employed. The velocity given by Equation 5 is a sum of sinusoidal components and the steady-state solution to Equation 1 with this u_b is,

$$u_w = \sum_{n=1}^{N} u_n \left\{ \frac{N_o\left(2\sqrt{nf\frac{z}{z_o}}\right)}{N_o(2\sqrt{nf})} \cos\left(n\omega t + \Phi_o\left(2\sqrt{nf\frac{z}{z_o}}\right) - \Phi_o(2\sqrt{nf}) \right) - \cos(n\omega t) \right\} \tag{6}$$

where $u_n = 2\pi n X(n)/L$. A wave described by stream function theory is uniquely defined by the two ratios h/L_o and H/L_o, where h is water depth, H wave height, and the subscript o denotes deepwater conditions. Waves with identical values on h/L_o and H/L_o yield the same dimensionless velocity $u_b/(H/T)$; thus, the quantity H/T appears as a normalizing "velocity". A wave friction factor (Jonsson 1980) derived for a stream function wave will depend not only on the normalized roughness k_n/H, but also on h/L_o and H/L_o. The friction velocity may be computed by using τ_{bav}, which is obtained from time integration of the absolute shear stress over a wave period T.

RESULTS

Measurements by Jonsson (1980) of u_w in a water tunnel for a sinusoidally varying u_b with the amplitude u_o was first employed to validate Equation 1 for describing the velocity in the TBL. It was verified that Equation 2 produced identical results to Equation 4, and the steady-state portion of the solution was used for the comparison with the data. Jonsson (1980) presented data for two cases: 1) $u_o = 2.11$ *m/s*, $T = 8.39$ *s*, $k_n = 2.3$ *cm*, and 2) $u_o = 1.53$ *m/s*, $T = 7.20$ *s*, $k_n = 6.3$ *cm*. Comparison between the analytical solution and the measurements was performed for the phases t/T = *1/2, 5/8, 3/4, 7/8,* and *1*. The Reynolds number Re for Cases 1 and 2 were $6.0\ 10^6$ and $2.7\ 10^6$, respectively, based on the bottom excursion amplitude A_b ($= u_o/\omega$) and u_o. The roughness values given by Jonsson were employed and there were no free calibration parameters. Two different definitions of u_{*m} were used in the comparison, namely $u_{*m} = (\tau_{bmax}/\rho)^{1/2}$ and $u_{*m} = (\tau_{bav}/\rho)^{1/2}$. Figures 2 and 3 display the comparison between the analytical solution and the measurements for Cases 1 and 2, respectively. In general, the difference between the two formulations for u_{*m} is small, although using τ_{bav} seems to consistently produce somewhat better agreement with the data. Some of the overshoot effect in the data is not entirely captured by the analytical solution, especially for Case 2.

Nadaoka et al. (1994) measured u_w in an oscillatory tunnel using air for a free-stream velocity that was asymmetric. The measurements used here to evaluate the TBL model involved a velocity u_b that was of cnoidal type with a positive peak velocity of 2.50 m/s, a negative peak velocity of 1.05 m/s, and a period of 5 s. A cnoidal wave producing a non-dimensional time variation in u_b corresponding to the experimental conditions implies an Ursell number of $U_r = 57.8$, although during the experiment u_b was generated to agree with the velocity induced by a hyperbolic wave. Such a strongly nonlinear wave provides a severe test for the linearized TBL equation; neglecting the nonlinear terms in the governing equation assumes that the particle velocity is small compare to the wave phase speed, which may not be the case for strongly nonlinear waves. However, for data obtained in oscillatory tunnels the spatial gradients should be small enough to permit that the nonlinear terms are neglected.

Instead of using a cnoidal or hyperbolic wave to describe u_b in the solution given by Equation 2, u_b was approximated using a wave described by 20-order stream function theory. Stream function, cnoidal, and hyperbolic theory could be employed to produce essentially identical variation in u_b with time, but the former theory allows direct calculation of u_w from Equation 6 for steady-state conditions without having to compute for the transient phase, which is necessary if the general solution in Equation 2 is employed. The bed consisted of spray-painted aluminum and was judged to be hydraulically smooth during the experiments. Thus, the length scale z_o is independent of the boundary roughness and may be calculated from $z_o = (3.3\nu_a/u_{*m})/30$, where ν_a is the kinematic viscosity for air. The air temperature was about 10 deg during the experiment and the corresponding Reynolds number was $Re = 2.8\ 10^5$. The origin of the vertical axis in the measurements was assumed to approximately coincide with z_o.

Since smooth turbulent flow prevailed during the experiment, z_o could be obtained from ν_a and u_{*m} and no calibration was needed to estimate the bed roughness. The representative shear velocity was based on τ_{bav}, which was determined through time integration over T. A value of $u_{*m} = 0.065$ m/s was thus calculated implying $z_o = 0.024$ mm. Figure 4 displays measured and calculated velocity profiles for selected phase values of t/T. The maximum positive peak in u_b occurred at about $0.18t/T$, the maximum negative peak at $0.68t/T$, and zero velocity at $0.36t/T$. The model captures the overall features of the velocity variation in the boundary layer, but the overshoot effect is not well predicted by the model, especially during the phase of flow reversal in the boundary layer in connection with large gradients in the wave velocity. The simple eddy viscosity model employed in the linearized TBL equation is most likely the reason for the disagreement between the model and the measurements, although lack of detailed information on z_o and the use of stream function theory to describe u_b may also contribute to the discrepancy.

REFERENCES

Brevik, I. 1981. "Oscillatory Rough Turbulent Boundary Layers," Journal of Waterways, Port, Coastal and Ocean Engineering, Vol 103, pp 175-188.

Dean, R.G. 1965. "Stream Function Representation of Nonlinear Ocean Waves," Journal of Geophysical Research, Vol 70, No. 18, pp 4561-4572.

Grant, W.D. and Madsen, O.S. 1979. "Combined Wave and Current Interaction with a Rough Bottom," Journal of Geophysical Research, Vol 84, No. C4, pp 1797-1808.

Grant, W.D. and Madsen, O.S. 1986. "The Continental-Shelf Bottom Boundary Layer," Annual Review of Fluid Mechanics, Vol 18, pp 265-305.

Jonsson, I. 1980. "A New Approach to Oscillatory Rough Turbulent Boundary Layers," Ocean Engineering, Vol 7, pp 109-152.

Kajiura, K. 1968. "A Model of the Bottom Boundary Layer in Water Waves," Bulletin of the Earthquake Research Institute, Vol 46, pp 75-123.

Myrhaug, D. 1982. "On a Theoretical Model of Rough Turbulent Wave Boundary Layers," Ocean Engineering, Vol 9, No. 6, pp 547-565.

Nadaoka, K., Yagi, H., Nihei, Y., and Nomoto, K. 1994. "Characteristics of the Structure of Turbulence Under Asymmetrical Oscillatory Flow," Proceedings of the Japanese Coastal Engineering Conference, Japan Society of Civil Engineers, pp 141-145. (in Japanese)

Nielsen, P. 1992. "Coastal Bottom Boundary Layers and Sediment Transport," World Scientific, Singapore.

Trowbridge, J. and Madsen, O.S. 1984. "Turbulent Wave Boundary Layers. 1. Model Formulation and First-Order Solution," Journal of Geophysical Research, Vol 89, No. C5, pp 7989-7997.

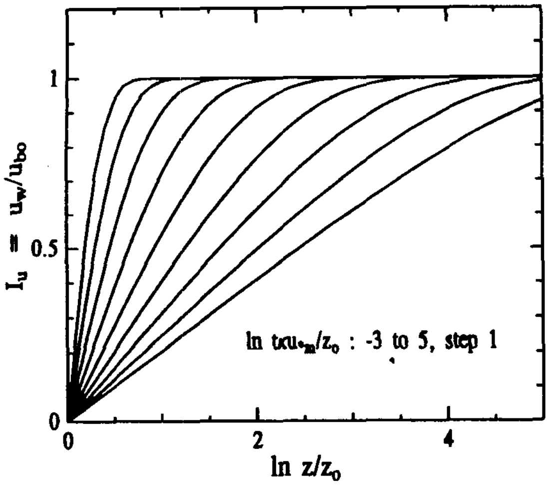

Fig. 1. The integral I_u as a function non-dimensional distance and time.

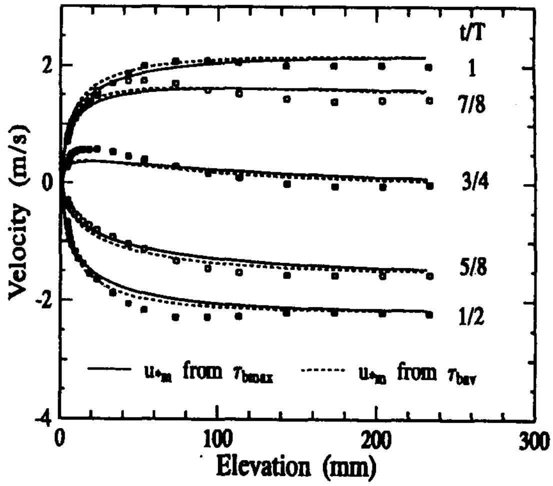

Fig. 2. Calculated and measured velocity in the turbulent boundary layer for Case 1 from the data by Jonsson.

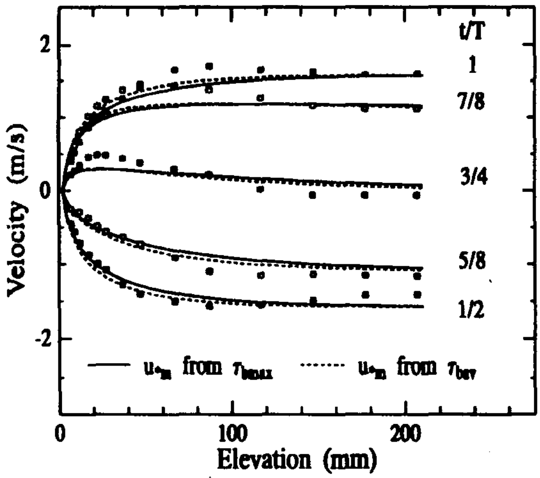

Fig. 3. Calculated and measured velocity in the turbulent boundary layer for Case 2 from the data by Jonsson.

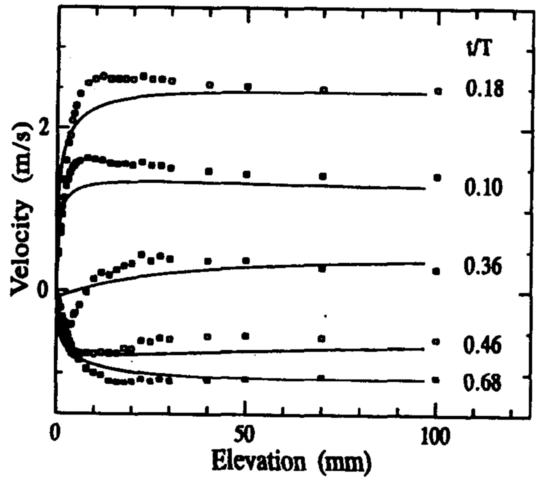

Fig. 4. Calculated and measured velocity in the turbulent boundary layer using the asymmetric velocity case from Nadaoka et al.