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The Fourier Analysis on DSA and P_2SA for Discrete-Ordinates Solutions of Neutron Transport Equations

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Abstract

By applying the P_1 and P_2 equations to the operator form of a synthetic acceleration, we derive the P_1-acceleration (diffusion synthetic acceleration: DSA) and P_2-acceleration (P_2SA) schemes in one dimensional slab geometry. We Fourier-analyze the derived acceleration schemes with the discrete-ordinates transport equation and showed that the DSA outperforms the P_2SA. These results confirm that one cannot simply assume that replacement of the DSA with a higher order approximation will lead to a better acceleration performance.

1. Introduction

Discrete-ordinates neutron transport codes use a dual iteration strategy for solving the transport equation. The two nested iteration loops are termed outer and inner iterations. The outer iteration represents a sweep through all the energy groups, while the inner iteration is performed within each energy group to update the flux-dependent neutron sources in the right-hand-side of the transport equation. Without an acceleration method in inner iterations, the discrete-ordinates transport solution is very slow, especially for optically thick regions and a scattering ratio near unity.[1] In a synthetic acceleration method[2] a low-order approximation to the transport operator is used to accelerate the slow convergence. After decades of work the DSA method [2,3], which uses a P_1 approximation, has emerged as a powerful and reliable tool for this purpose.

Recent research result shows that the P_2 (Simplified P_2 in multi-dimensional cases) equation[4] is nearly always more accurate than the diffusion equation with essentially the same amount of computational effort. Intuitively, therefore, one might expect that replacing the P_1 approximation with P_2 lead to an improved acceleration of the transport iteration. It should be emphasized that our analysis is based on the continuous form of P_1,

P_2, and the transport equation. The computational errors caused by spatial or angular discretizations are not considered here.

2. Formulation of P_1 and P_2 Equations

The derivation of the P_1 and P_2 equations in slab geometry appears elsewhere.[1,4] For completeness, however, the derivation will be briefly repeated. Using conventional notation, the with-in group, first-order transport equation with isotropic scattering and source in slab geometry is

$$\mu\frac{\partial\varphi}{\partial x}+\sigma\varphi(x,\mu)=\sigma_s\phi_0(x)+Q\ . \tag{1}$$

The expansion of the angular flux φ in the first N+1 Legendre polynomial,

$$\varphi(x,\mu)=\sum_{n=0}^{N}(2n+1)\phi_n(x)P_n(\mu) \tag{2}$$

is referred to as the P_N method and the transport equation [Eq.(1)] is approximated by the following P_N equations:

$$\frac{n}{2n+1}\frac{d\phi_{n-1}}{dx}+\frac{n+1}{2n+1}\frac{d\phi_{n+1}}{dx}+\sigma\phi_n=\delta_{n0}\left(\sigma_s\phi_n+Q\right),\quad n=0,1,2,\cdots,N \tag{3}$$

with δ_{n0}, the Kronecker delta. The Legendre moments in Eq.(3) are given by

$$\phi_n(x)=\frac{1}{2}\int_{-1}^{1}d\mu P_n(\mu)\varphi(x,\mu)\ . \tag{4}$$

In the P_N approximation, we truncate Eq.(3) by assuming $\frac{d\phi_n}{dx}=0$ for $n>N$. If we apply a P_1 approximation to Eq.(3), we have the following two equations

$$\frac{d\phi_1}{dx}+\sigma_a\phi_0=Q \qquad \frac{1}{3}\frac{d\phi_0}{dx}+\sigma\phi_1=0 \tag{5}$$

Solving Eqs.(5) for ϕ_0 gives the P_1 (diffusion) equation:

$$-\frac{d}{dx}D\frac{d\phi_0}{dx}+\sigma_a\phi_0=Q\ , \tag{6}$$

where D is the diffusion coefficient defined as $1/3\sigma$.

Similar procedure with P_2 approximation to Eq.(3) gives a P_2 equation

$$-\frac{d}{dx}D\frac{d}{dx}\left[\phi_0+\frac{4}{5\sigma}\left(\sigma_a\phi_0-Q\right)\right]+\sigma_a\phi_0=Q\ , \tag{7}$$

which is also a single diffusion equation.

3. Formulation of DSA and P_2SA Equations

The synthetic acceleration involves solving alternate transport and low-order equation. However, to get a convergent solution, a lower-order equation should be modified to couple the two different equations and it can be achieved by adding the source correction terms in a lower-order equation. In the conventional P_N transport acceleration,[3] an acceleration is achieved by solving the P_N equations in which the $(N+1)$'th Legendre moment is known from the previous iteration and is treated as a source. In our approach developed elsewhere[5], however, the acceleration equations are provided by inserting the regular P_N equation -which is truncated at N'th moment- into an operator form of synthetic acceleration. Because of the lack of space, we just write down the final form of synthetic acceleration:

$$H_L \phi_0^{\ell+1} = Q - \int d\hat{\Omega}(H - H_L)\varphi^{\ell+1/2} \ , \tag{8}$$

where H is the transport operator and H_L is any lower-order operator which is independent of $\hat{\Omega}$. In this work, we use the even-parity equation[1] as the transport operator. The choice of the form of transport equation does not change our development, since they are the same equations as long as the variables are treated continuously. The even-parity equation, owing to its second-order differential operator, facilitates the derivation of DSA and P_2SA.

To derive a DSA scheme we take the P_1 (diffusion) operator [Eq.(6)] as a low-order operator H_L :

$$H_L \phi_0^{\ell+1} = -\frac{d}{dx} D \frac{d\phi_0^{\ell+1}}{dx} + \sigma_a \phi_0^{\ell+1} \ , \tag{9}$$

and the even-parity operator as a high-order operator H:

$$H\chi^{\ell+1/2} = -\mu^2 \frac{\partial}{\partial x} \frac{1}{\sigma} \frac{\partial \chi^{\ell+1/2}}{\partial x} + \sigma \chi^{\ell+1/2} - \sigma_s \phi_0^{\ell+1/2} \ , \tag{10}$$

where χ is an even-parity angular flux.

Inserting Eqs.(9) and (10) into Eq.(8) yields

$$-\frac{d}{dx} D \frac{d\phi_0^{\ell+1}}{dx} + \sigma_a \phi_0^{\ell+1} = Q + 2\frac{d}{dx} D \frac{d\phi_2^{\ell+1/2}}{dx} \ , \tag{11}$$

where

$$\phi_2^{\ell+1/2} = \int_0^1 d\mu P_2(\mu) \chi^{\ell+1/2} \ .$$

Equation (11) happens to be the same as the conventional DSA scheme, so the effectiveness of acceleration would be the same.

For a P_2SA, we take the P_2 operator [Eq.(7)] as a low-order operator H_L :

$$H_L \phi_0^{\ell+1} = -\frac{d}{dx} D \frac{d}{dx}\left[\phi_0^{\ell+1} + \frac{4}{5\sigma}\left(\sigma_a \phi_0^{\ell+1} - Q\right)\right] + \sigma_a \phi_0^{\ell+1} \ . \tag{12}$$

and use the same even-parity operator [Eq.(10)] as an H. Similarly, substituting Eqs.(12) and (10) into Eq.(8) yields

$$-\frac{d}{dx} D \frac{d}{dx}\left(1 + \frac{4\sigma_a}{5\sigma}\right)\phi_0^{\ell+1} + \sigma_a \phi_0^{\ell+1} = Q + 2\frac{d}{dx} D \frac{d\phi_2^{\ell+1/2}}{dx} - \frac{d}{dx} D \frac{d}{dx}\left(\frac{4\sigma_a}{5\sigma}\right)\phi_0^{\ell+1/2} \ , \tag{13}$$

where

$$\phi_0^{\ell+1/2} = \int_0^1 d\mu \chi^{\ell+1/2} .$$

Equation (13) is a new P_2SA scheme in which the second Legendre moment appears as the highest moment term. The conventional P_2-acceleration has the third Legendre moment as a source correction term.

4. Fourier Analysis

In this section, we analyze the convergence rate of each acceleration scheme by performing a Fourier analysis. We will recast the transport and acceleration equations to amenable forms by defining the variables

$$x = \int_0^z dz'\sigma(z') , \quad c = \sigma_s / \sigma , \quad S = Q / \sigma .$$

Then the residual form of the transport equation for the successive iteration ℓ is

$$-\mu^2 \frac{\partial^2 \tilde{\chi}^{\ell+1/2}}{\partial x^2} + \tilde{\chi}^{\ell+1/2} = c\tilde{\phi}_0^{\ell} , \tag{14}$$

where

$$\tilde{\chi}^{\ell+1/2} = \chi^{\ell+1/2} - \chi^{\ell-1/2} ,$$

and

$$\tilde{\phi}_0^{\ell} = \phi_0^{\ell} - \phi_0^{\ell-1} .$$

Similarly, for the DSA equation, we get

$$-\frac{1}{3}\frac{d^2\tilde{\phi}_0^{\ell+1}}{dx^2} + (1-c)\tilde{\phi}_0^{\ell+1} = \frac{2}{3}\frac{d^2\tilde{\phi}_2^{\ell+1/2}}{dx^2} , \tag{15}$$

where

$$\tilde{\phi}_2^{\ell+1/2} = \phi_2^{\ell+1/2} - \phi_2^{\ell-1/2} .$$

Now, Eqs.(14) and (15) are homogeneous differential equations so they are separable with repect to the spatial and angular variables, x and μ. We may express the errors between iteration steps as Fourier series as

$$\tilde{\chi}^{\ell+1/2} = \sum_{k=-\infty}^{\infty} \omega_k^{\ell} a_k(\lambda_k,\mu) e^{i\lambda_k x} , \tag{16a}$$

$$\tilde{\phi}_0{}^{\ell} = \sum_{k=-\infty}^{\infty} \omega_k^{\ell} b_k(\lambda_k) e^{i\lambda_k x} , \tag{16b}$$

$$\tilde{\phi}_2^{\ell+1/2} = \int_0^1 d\mu P_2(\mu)\tilde{\chi}^{\ell+1/2} = \sum_{k=-\infty}^{\infty} \omega_k^{\ell} e^{i\lambda_k x} \int_0^1 d\mu P_2(\mu) a_k(\lambda_k,\mu) , \tag{16c}$$

where ω_k is a function of the parameter $\lambda_k = \frac{k\pi}{L}$ (L is the domain of x), and $i = \sqrt{-1}$.

The propagation of error is obviously governed by the type of equation and the convergence is studied by noting the behavior of ω_k for the k'th harmonic. We expect that the corresponding error harmonic would grow beyond limit for increasing ℓ for $|\omega_k|>1$, and that the error would shrink for increasing ℓ for $|\omega_k|<1$. Substituting those expansions in Eqs.(16) into Eqs.(14) and (15), we easily obtain ω of the k'th harmonic for DSA

$$\omega_{\rm DSA} = \frac{-2\lambda^2 c\int_0^1 d\mu \frac{P_2(\mu)}{1+\lambda^2\mu^2}}{\lambda^2+3(1-c)} , \qquad (17)$$

a well known result.[2]

Similarly, the second moment P_2SA equation can be obtained by subtracting Eq.(13) for successive values of ℓ, yielding

$$-\frac{1}{3}\frac{d^2}{dx^2}\left(\frac{9-4c}{5}\right)\tilde{\phi}_0^{\ell+1} + (1-c)\tilde{\phi}_0^{\ell+1} = \frac{2}{3}\frac{d^2\tilde{\phi}_2^{\ell+1/2}}{dx^2} - \frac{4}{15}\frac{d^2}{dx^2}(1-c)\tilde{\phi}_0^{\ell+1/2} . \qquad (18)$$

Inserting the same expansions in Eqs.(16) and

$$\tilde{\phi}_0^{\ell+1/2} = \int_0^1 d\mu P_0(\mu)\tilde{\chi}^{\ell+1/2} = \sum_{k=-\infty}^{\infty}\omega_k^{\ell} e^{i\lambda_k x}\int_0^1 d\mu a_k(\lambda_k,\mu)$$

into Eqs.(14) and (18), we find that

$$\omega_{\rm P_2SA} = \frac{-2\lambda^2 c\int_0^1 d\mu \frac{P_2(\mu)}{1+\lambda^2\mu^2} + \frac{4}{5}c(1-c)\lambda^2\int_0^1 \frac{d\mu}{1+\lambda^2\mu^2}}{\left(\frac{9-4c}{5}\right)\lambda^2 + 3(1-c)} . \qquad (19)$$

For the k'th harmonic, the ω of DSA and P_2SA given by Eqs.(17) and (19) are plotted against the parameter λ in Fig.1. When the scattering ratio c is unity, the P_2SA becomes exactly same as DSA. For $0<c<1$, however, the ω's of DSA are always smaller than those of P_2SA, which means DSA converges faster than P_2SA in accelerating the transport solution in practical problems.

5. Conclusions

Any acceleration equation which is derived from the operator form of synthetic acceleration is the angularly integrated transport equation with no approximation. Therefore, the order of P_N approximation in a low-order operator does not make any difference in accuracy of the corresponding acceleration equation as long as the source correction terms are provided by the transport solutions. The acceleration performance depends only on the parameter ω which is governed by the specific type of the acceleration equation.

References

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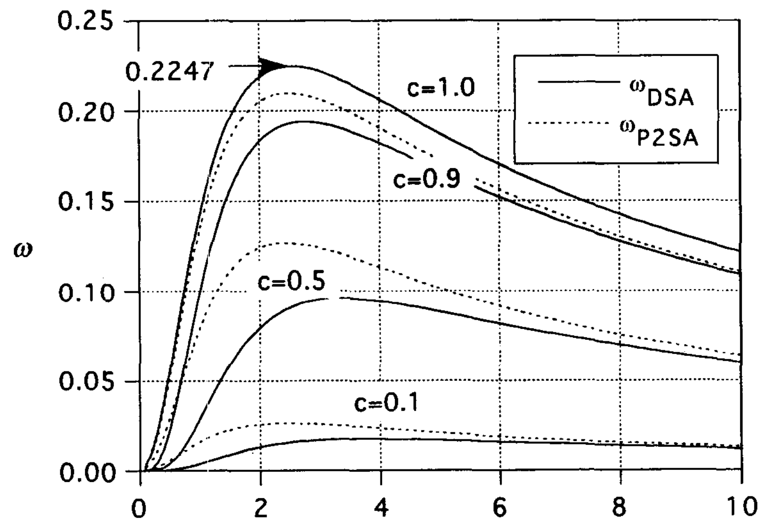

Fig.1. ω of DSA and P_2SA