탄소성 대변형에 관한 비등방 구성방정식
Anisotropic Constitutive Model at Large Viscoplastic Deformations

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ABSTRACT

A new combined isotropic/kinematic and orthotropic hardening viscoplastic model is proposed which can account for not only differential orientations but also preferred orientations of grains in a metal at finite plastic deformations with an introduction of multiple spin (rate of rotation) concept. Within the general framework of the model, the effects of anisotropy and constitutive spins will be discussed in conjunction with a closed-form solution for simple shear in a rigid-plastic material, which will be used to simulate experimental data of Montheillet, et al. (1984) for fixed-end tortion tests at finite plastic deformations.

1. INTRODUCTION

Initial random orientation of grains in a metal tends to acquire a directional property with plastic deformations. Bauschinger effect, which is a macroscopic phenomenon of differential orientations of grains with deformations in a metal, can be accounted for by kinematic hardening model in terms of internal stress (back stress). However, as the plastic deformations of metal become finite, preferred orientations of grains may occur, which cannot be explained by kinematic hardening model. Although a plethora of theoretical analyses for the behavior of metals at finite inelastic deformations have been presented by many researchers, these two phenomena have not been mentioned together in one constitutive model and little has been done for comparison between theory and experiments. With the motivation of Hill's idea (1950) of material transition from initial isotropy to later anisotropy with finite plastic deformations, a new combined isotropic/kinematic and orthotropic hardening model is proposed and simple shear is analyzed to simulate the experimental data of Montheillet, et al. (1984).

In contemorary theories of elastoplasticity at finite strain, the concept of the plastic spin (the plastic rate of rotation), which is the difference of the spin of continuum from its underlying substructure, seems to be necessary. Since Mandel (1971) and Kratochvil (1971) originally proposed the concept for the plastic spin, the study for general finite deformation plasticity

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with the plastic spin has been carried out by Dafalias (1985, 1990), Loret (1983) and Dafalias and Cho (1989) followed by many researchers. Based on the idea of Dafalias and Cho (1989), this paper discusses the transition of a material from initial isotropy to orthotropy with finite viscoplastic deformations within the framework of a new proposed hardening model with multiple plastic spins.

Tensors will be denoted by boldface characters in direct notation. With the summation convention over repeated indices, the following symbolic operations are implied: $\mathbf{a}\boldsymbol{\sigma} = a_{ij}\sigma_{jk}$, $\mathbf{a}:\boldsymbol{\sigma} = a_{ij}\sigma_{ji}$, $\mathbf{a}\otimes\boldsymbol{\sigma} = a_{ij}\sigma_{kl}$, with proper extensions to the tensors of different order. The prefix tr indicates the trace, and a superposed dot denotes the material time derivative or rate.

2. KINEMATICS AND KINETICS IN VISCOPLASTICITY AT LARGE DEFORMATIONS

The introduction of the concept of intermediate (unstressed) configuration by Lee (1969) and the director vectors by Mandel (1971) leads the kinematics in small elastic and finite plastic deformations to be expressed as follows:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \tag{1}$$

$$\mathbf{W} = \boldsymbol{\omega} + \mathbf{W}^p = \boldsymbol{\omega}_c + \mathbf{W}^p_c \tag{2}$$

where $\mathbf{D}$, $\mathbf{D}^e$ and $\mathbf{D}^p$ denote the total, elastic and plastic rate of deformations, respectively, and $\mathbf{W}$, $\boldsymbol{\omega}$ and $\mathbf{W}^p$ the total, substructural and plastic spins, respectively, in Eulerian kinematics. The $\boldsymbol{\omega}_c$ denotes a representative consitutive spin used in the rate equation of evolution of internal variables. The $\mathbf{W}^p_c$ is the difference of $\mathbf{W}$ and $\boldsymbol{\omega}_c$. The material state will be defined at the current configuration in terms of the Cauchy stress $\boldsymbol{\sigma}$ and a representative collection $\mathbf{a}$ of structure variables. It is assumed that the $\mathbf{a}$, which in fact provides anisotropic properties via their variations (Onat, 1982), be elastically embedded in the continuum. The corotational rates for $\boldsymbol{\sigma}$ and $\mathbf{a}$ with respect to $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_c$, respectively, are defined as

$$\tilde{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\omega}\,\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\omega}, \qquad \check{\mathbf{a}} = \dot{\mathbf{a}} - \boldsymbol{\omega}_c\mathbf{a} + \mathbf{a}\boldsymbol{\omega}_c \tag{3}$$

Based on Dafalias (1990), kinetics can be described as follows:

$$\mathbf{D}^e = \mathbf{L}^{-1}:\ \tilde{\boldsymbol{\sigma}} + \mathbf{T}\,\dot{\theta} \tag{4}$$

$$\mathbf{D}^p = \langle\Phi\rangle\ \mathbf{N}^p\,, \qquad \mathbf{W}^p = \langle\Phi\rangle\boldsymbol{\Omega}^p \tag{5}$$

$$\check{\mathbf{a}} = \langle\Phi_i\rangle\ \bar{\mathbf{a}}_i \tag{6}$$

where $\mathbf{L}$ represents the elastic moduli, $\mathbf{T}$ is the thermal expansion tensor, θ is the temperature, and Φ is non-negative scalar-valued overstress function, and $\mathbf{N}^p$, $\boldsymbol{\Omega}^p$ and $\bar{\mathbf{a}}_i$ define the "direction" of $\mathbf{D}^p$, $\mathbf{W}^p$, $\check{\mathbf{a}}$, respectively. The $\langle\rangle$ is the Macauley bracket. Invariance requirements under superposed rigid body rotation render all tensor and scalar valued constitutive functions of Eqs. (4), (5) and (6) isotropic functions of their variables $\boldsymbol{\sigma}$, θ and $\mathbf{a}$.

3. NEW COMBINED HARDENING MODEL

A viscoplastic formulation is proposed which can account for both isotropic/kinematic hardening and orthotropic symmetries with the axes of orthotropy along the unit vectors $\mathbf{n}_i$. Denoting by superposed ^ the tensor components in reference to the orthotropic $\mathbf{n}_i$-axes, and by $\boldsymbol{\sigma}$, s and $\boldsymbol{\alpha}$, the Cauchy stress, the deviatoric Cauchy stress and the deviatoric back stress tensors, respectively, and introducing $\boldsymbol{\sigma}^* = \boldsymbol{\sigma} - \boldsymbol{\alpha}$, one can define the quantity

$$
\begin{aligned}
J &\equiv \{A(\hat{\sigma}^*_{11} - \hat{\sigma}^*_{22})^2 + B(\hat{\sigma}^*_{22} - \hat{\sigma}^*_{33})^2 + C(\hat{\sigma}^*_{33} - \hat{\sigma}^*_{11})^2 \\
&\quad + 2D\,\hat{\sigma}^{*2}_{23} + 2E\,\hat{\sigma}^{*2}_{31} + 2F\,\hat{\sigma}^{*2}_{12}\}^{1/2} \\
&= \{(A+B+4C-2E)\,\mathrm{tr}^2(\mathbf{a}_1\mathbf{s}^*) + (A+4B+C-2D)\,\mathrm{tr}^2(\mathbf{a}_2\mathbf{s}^*) \\
&\quad + 2(-A+2B+2C-D-E+F)\,\mathrm{tr}(\mathbf{a}_1\mathbf{s}^*)\,\mathrm{tr}(\mathbf{a}_2\mathbf{s}^*) \\
&\quad + 2(F-D)\,\mathrm{tr}(\mathbf{a}_1\mathbf{s}^{*2}) + 2(F-E)\,\mathrm{tr}(\mathbf{a}_2\mathbf{s}^{*2}) + (D+E-F)\,\mathrm{tr}(\mathbf{s}^{*2})\}^{1/2}
\end{aligned}
\tag{7}
$$

where $\mathbf{a}_i = \mathbf{n}_i \otimes \mathbf{n}_i$, i=1,2,3, (not summation), and A, B, C, D, E and F are the material parameters. For the case of $\boldsymbol{\alpha} = \mathbf{0}$, Eq. (7) produces the orthotropic hardening model of Hill (1950), while for A=B=C=1 and D=E=F=3 this reduces to von Mises' type kinematic hardening with $J = \sqrt{3}\{\mathrm{tr}(\mathbf{s}-\boldsymbol{\alpha})^2\}^{1/2}$ for an isotropic material.

A power law overstress function is assumed as follows:

$$\Phi = [(J - \sqrt{2}k)/V]^n \tag{8}$$

$$\mathbf{D}^p = \langle\Phi\rangle\,(\partial J/\partial \boldsymbol{\sigma}) \tag{9}$$

where V and n are the material parameters, and k is the size of static yield surface. With the introduction of equivalent plastic strain $\bar{\varepsilon}^p$ and the rate $\dot{\bar{\varepsilon}}^p = [(2/3)\mathbf{D}^p{:}\mathbf{D}^p]^{1/2}$, Eqs. (8) and (9) yield dynamic yield surface as:

$$
\begin{aligned}
J(\boldsymbol{\sigma},\mathbf{a}_1,\mathbf{a}_2) &= \sqrt{2}k + V[\{(2/3)\,\mathrm{tr}(\partial J/\partial \boldsymbol{\sigma})^2\}^{-1/2}\,\dot{\bar{\varepsilon}}^p)]^{1/n} \\
&= \sqrt{2}k + V^*\,(\dot{\bar{\varepsilon}}^p)^{1/n} \equiv \delta
\end{aligned}
\tag{10}
$$

where V^* is a new material parameter which incorporates V and $\mathrm{tr}(\partial J/\partial \boldsymbol{\sigma})$. Eq. (10) shows the dependence of J on the strain rate $\dot{\bar{\varepsilon}}^p$.

In summary, four important concepts of this model can be described as follows:

(i) k, size of yield surface portraying isotropic hardening

$$k = k_o + R, \qquad k_o\text{: initial value of k}, \quad R\text{: variable part} \tag{11a}$$

$$\dot{R} = (H - C_r R)\,\dot{\bar{\varepsilon}}^p - C_s R \qquad H, C_r, C_s\text{: material parameters} \tag{11b}$$

(ii) $\boldsymbol{\alpha}$, deviatoric back stress portraying kinematic hardening

$$\overset{\vee}{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} - \boldsymbol{\omega}_k\,\boldsymbol{\alpha} + \boldsymbol{\alpha}\boldsymbol{\omega}_k = (2/3)h_a\mathbf{D}^p - c_r(\dot{\bar{\varepsilon}}^p)^s\boldsymbol{\alpha} - c_s\boldsymbol{\alpha} \tag{12a}$$

h_a, c_r, c_s, s: material parameters

$$\boldsymbol{\omega}_k = \mathbf{W} - \mathbf{W}^p_k \tag{12b}$$

$$\mathbf{W}^p_k = \langle\Phi\rangle\boldsymbol{\Omega}^p_k = \langle\Phi\rangle(1/2)\rho(\boldsymbol{\alpha}\,\partial J/\partial\boldsymbol{\sigma} - \partial J/\partial\boldsymbol{\sigma}\,\boldsymbol{\alpha}) = (1/2)\rho(\boldsymbol{\alpha}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\alpha}) \tag{12c}$$

(iii) A,B,C,D,E,F, material constants portraying distortional hardening

(iv) $\mathbf{n}_i$, orthotropic axes portraying directional hardening

$$\overset{\vee}{\mathbf{n}}_i = \dot{\mathbf{n}}_i - \boldsymbol{\omega}_o\mathbf{n}_i = \mathbf{0} \tag{13}$$

$$\boldsymbol{\omega}_o = \mathbf{W} - \mathbf{W}^p_o = x\boldsymbol{\Omega}^E + (1-x)(\mathbf{W} - \overline{\mathbf{W}}^p_o)$$

$$\mathbf{W}^{p}{}_{o} = \langle\Phi\rangle[\eta_1(a_1\sigma-\sigma a_1) + \eta_2(a_2\sigma-\sigma a_2) + \eta_3(a_1\sigma a_2-a_2\sigma a_1)] \quad (14)$$

where ω_o is the substructural spin for $\mathbf{n}_i$, Ω^E the Eulerian spin, η_i the material parameters, and x is the scalar-valued transition coefficient depending on deformations.

4. APPLICATION TO SIMPLE SHEAR

The motion for simple shear in x_1-x_2 plane is expressed by

$$x_1 = X_1 + \gamma(t)\, X_2$$
$$x_2 = X_2$$
$$x_3 = X_3 \quad (15)$$

where x_i and X_i, i=1,2,3, are Cartesian coordinates of current and initial position of a material point, respectively, and γ is the shear strain. All components with a superposed ^ will be referred to the orthotropic axes $\mathbf{n}_i$. The velocity gradient components are

$$D_{12} = D_{21} = W_{12} = -W_{21} = \dot{\gamma}/2$$
$$D_{ij} = W_{ij} = 0 \quad \text{for other i, j combinations.} \quad (16)$$

Since this work is concerned with large plastic deformations, a rigid-plastic material response will be for simplicity assumed, i.e.:

$$\mathbf{D}^{p} = \mathbf{D}. \quad (17)$$

4.1 Analysis of Simple Shear

With Eqs. (7), (8) and (9) in $\mathbf{n}_i$-axes and the transformation of the rate of deformation conponents from $\mathbf{x}_i$ to $\mathbf{n}_i$-axes, it is straightforward to obtain non-zero stress components as follows:

$$(\hat{\sigma}^*_{11} - \hat{\sigma}^*_{33})/B = -(\hat{\sigma}^*_{22} - \hat{\sigma}^*_{33})/C = \hat{\sigma}^*_{12}\, F\tan(2\Phi)/X = \mathrm{sgn}(\dot{\gamma}\sin 2\Phi)J/R \quad (18)$$

with X = AB + BC + CA

$$R = [X\{B + C + 2X/(F\tan^2 2\Phi)\}]^{1/2}$$

Assuming that $\hat{\sigma}_{33} = \sigma_{33} = 0$ and $\hat{\alpha}_{33} = \alpha_{33} = 0$, the transformation of Eq. (18) from n_i to x_i coordinate system yields

$$\frac{(\sigma_{11}-\alpha_{11})}{J} = sgn(\dot{\gamma}\sin 2\phi)\frac{1}{R}\left[\frac{B-C}{2} + \left(\frac{B+C}{2} - \frac{X}{F}\right)\cos 2\phi\right]$$

$$\frac{(\sigma_{22}-\alpha_{22})}{J} = sgn(\dot{\gamma}\sin 2\phi)\frac{1}{R}\left[\frac{B-C}{2} - \left(\frac{B+C}{2} - \frac{X}{F}\right)\cos 2\phi\right]$$

$$\frac{(\sigma_{12}-\alpha_{12})}{J} = sgn(\dot{\gamma})\,|\sin 2\phi|\,\frac{1}{R}\left[\frac{B+C}{2} + \frac{X}{F\tan^2 2\phi}\right] \quad (19)$$

with the system of differential equations of α_{11} and α_{12}, based on Eq. (12a), as

$$\frac{d\alpha_{11}}{d\gamma} = -sgn(\dot{\gamma})\frac{1}{\sqrt{3}}\, c_k\alpha_{11} + (1-\rho\alpha_{11})\alpha_{12}$$

$$\frac{d\alpha_{12}}{d\gamma} = -sgn(\dot{\gamma})\frac{1}{\sqrt{3}}\, c_k\alpha_{12} - (1-\rho\alpha_{11})\alpha_{11} + \frac{1}{3}h_a$$

$$c_k = c_r\,(\dot{\bar{\varepsilon}})^{s-1} + \frac{c_s}{\dot{\bar{\varepsilon}}} \quad (20)$$

where $\dot{\bar{\varepsilon}}^p \cong \dot{\bar{\varepsilon}} = \frac{|\dot{\gamma}|}{\sqrt{3}}$ for monotonic change of γ. For the particular case of cubic orthotropic symmetries where A=B=C=a and F=b, Eqs. (19) become

$$-\frac{(\sigma_{11}-\alpha_{11})}{J} = \frac{(\sigma_{22}-\alpha_{22})}{J}$$

$$= sgn(\dot{\gamma}\cos 2\phi)(\frac{3a}{b}-1)\frac{\sin 2\phi}{[\ 6a(\frac{3a}{b}+\tan^2 2\phi)]^{\ 1/2}}$$

$$\frac{(\sigma_{12}-\alpha_{12})}{J} = sgn(\dot{\gamma})[\ \frac{1}{6a}[\ 1+(\frac{3a}{b}-1)\cos^2 2\phi\ ]\ ]^{\ 1/2} \qquad (21)$$

It is noted that where a=1 and b=3, Eqs. (7) reduce to von Mises' type kinematic hardening criterion, i.e. $(\sigma_{12}-\alpha_{12})/J = sgn(\dot{\gamma})/\sqrt{6}$ for an isotropic material, independent of ϕ. For the variation of the parameters a and b, the followings are proposed:

$$a = (1-a_s)\exp(-c_a\ |\gamma|\ ) + a_s$$

$$b = (1-b_s)\exp(-c_b\ |\gamma|\ ) + b_s \qquad (22)$$

where a_s and b_s are the saturated values of a and b with deformations, and c_a and c_b are the material constants. The parameters of Eq. (14), which portray the transition from isotropy to orthotropy, are proposed as:

$$x = \exp(-c\ |\gamma|\ )$$

$$\eta = \frac{\delta}{F}(\eta_1-\eta_2+\eta_3) \qquad (23)$$

where c is a material constant.

Finally, the dynamic yield surface of Eq. (7) reduces to

$$J^2 = (6a-3b)[tr^2(a_1s^*) + tr^2(a_2s^*) + tr(a_1s^*)\ tr(a_2s^*)] + b\ tr(s^{*2}) = \delta^2 \qquad (24)$$

From Eq. (21), $\sigma^*_{11} + \sigma^*_{22} = 0$, (25)
hence, one can obtain the following criterion, which is in fact the projection of the intersection of the yield surface with the plane of Eq. (24), onto the $\sigma^*_{11} - \sigma^*_{12}$ plane, as follows:

$$J^2 = [2b + (6a-2b)\cos^2 2\phi]\ \sigma^{*2}_{11} + [2b + (6a-2b)\sin^2 2\phi]\ \sigma^{*2}_{12}$$
$$+ (6a-2b)\sin 4\phi\ \sigma^*_{11}\sigma^*_{12} = \delta^2 \qquad (26)$$

Note that for a=1 and b=3 Eq. (25) yields $\sigma^{*2}_{11}+\sigma^{*2}_{12} = (\sigma_{11}-\alpha_{11})^2+(\sigma_{12}-\alpha_{12})^2 = \delta^2/6$ for isotropy.

4.2 Simulation of Experimental Data

Fixed-end tortional experiments of Montheillet, et al. (1984) for α-Fe were examined with the parameters of Sec. 4.1 within the analysis of simple shear. Although the details are omitted due to the limited space, Fig. 1 shows satisfactory simulation for strain rate effects on the stresses and the phenomena of variation of axial stress σ_{22} from negative to positive quantity with deformations. The transition of the yield surface is clearly seen in Fig. 2, where the path of the current stress point σ is explained with the transition of the yield surface. The rotation or orientation of the surface can be discussed via n_i or ϕ of Eqs. (13), (14) and (23) while the distortion is related with the parameters, a and b, of Eqs. (21) and (22).

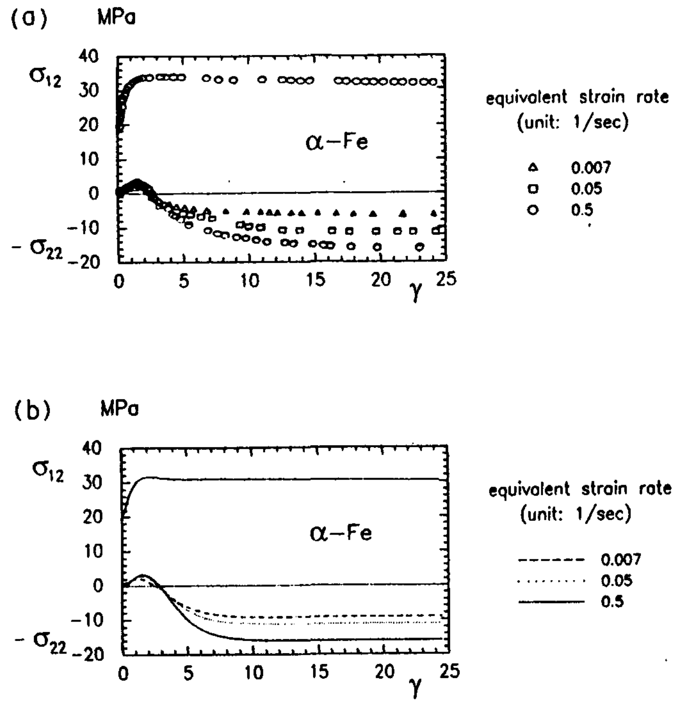

Figure 1 (a) Experimental stress/strain curves for α-Fe at 800 °C and different strain rates after Montheillet, et al. (1984), and (b) Simulation of the stress/strain curves for α-Fe

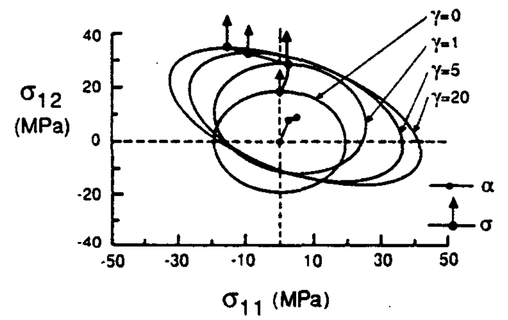

Figure 2 Evolution of the dynamic yield surface and associated stress path

5. CONCLUSION

Residual stress and texture development are indispensable parts in the constitutive formulation for finite inelastic deformations of metals. These two different phenomena were implemented within the framework of the proposed hardening model, which could accomplish successful simulation for the experimental data of simple shear by means of the distortional and orientational features of yield surface.

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