A NOTE ON a-FUZZY CLOSED AND a-FUZZY CONTINUOUS MAPPINGS

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0. Introduction.

We introduce new weak forms of fuzzy continuity and fuzzy closed mapping(which we call a-fuzzy continuity and a-fuzzy closed mapping). And we investigate some of the basic properties of a-fuzzy continuous mappings and a-fuzzy closed mappings.

1. Preliminaries.

The symbols X, Y and Z denote fuzzy topological spaces with no separation axioms assumed unless explicitely stated. The closure, interior and complement of a fuzzy set A in a fuzzy topological space X are denoted by $Cl(A)$, $Int(A)$, and $\mathbb{C}A$, respectively.

Definition1.1[2]. *A fuzzy set A in a fuzzy space X is said to be* **g-fuzzy closed** *if $Cl(A) \subset U$ whenever $A \subset U$ and U is fuzzy open in X. A fuzzy set A is said to be g-fuzzy open in X if $\mathbb{C}A$ is g-fuzzy closed in X.*

Theorem A[2]. *A fuzzy set A is g-fuzzy open if and only if $F \subset Int(A)$ whenever F is fuzzy closed and $F \subset A$.*

Definition 1.2[2]. *A mapping $f : X \to Y$ is said to be* **g-fuzzy continuous** *if for each closed fuzzy set F in Y, $f^{-1}(F)$ is g-fuzzy closed in X.*

2. a-Fuzzy Closed and a-Fuzzy Continuous Mappings.

Definition 2.1. *Let X and Y be fuzzy topological spaces. Then:*

(1) A mapping $f : X \to Y$ is said to be **approximately fuzzy closed***(or simply* **a-fuzzy closed***) if $f(F) \subset Int(A)$ whenever F is a closed fuzzy set in X, A is a g-open fuzzy set in Y, and $f(F) \subset A$.*

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(2) A mapping $f : X \to Y$ is said to be **approximately fuzzy continuous** *(or simply* **a-fuzzy continuous***) if $Cl(A) \subset f^{-1}(V)$ whenever V is an open fuzzy set in Y, A is a g-closed fuzzy set in X, and $A \subset f^{-1}(V)$.*

It is clear that fuzzy closed mappings are a-fuzzy closed, and fuzzy continuous mappings are a-fuzzy continuous. The following Example shows that converse implications do not hold.

Example 2.2. *Let $X = \{a, b\}$ and let $\tau = \{\underline{0}, \{(a, \lambda), (b, 0)\}, \{(a, 0), (b, \mu)\}, \{(a, \lambda), (b, \mu)\}, \underline{1}\}$, where $\lambda \neq \mu$ and $0 < \lambda, \mu \leq 1$. Then clearly τ is a fuzzy topology on X. Let $f : X \to X$ be the mapping defined by $f(a) = b$, and $f(b) = a$. Then f is neither fuzzy closed nor fuzzy continuous. However, since the image of each closed fuzzy set is fuzzy open, f is a-fuzzy closed. Also since the inverse image of each open fuzzy set is fuzzy closed, f is a-fuzzy continuous.*

The proof of the following result is a straightforward argument using complements and is omitted.

Theorem 2.3. *Let $f : X \to Y$ be bijective. Then f is a-fuzzy closed if and only if f^{-1} is a-fuzzy continuous.*

3. Preserving g-Closed Fuzzy Sets.

Theorem 3.1. *If $f : X \to Y$ is g-fuzzy continuous and a-fuzzy closed, then $f^{-1}(A)$ is g-fuzzy closed(or g-fuzzy open) in X whenever A is g-fuzzy closed(or g-fuzzy open) in Y.*

Proof. Let A be a g-closed fuzzy set in Y and let U be any open fuzzy set in X such that $f^{-1}(A) \subset U$. Then $\complement U \subset \complement f^{-1}(A) = f^{-1}(\complement A)$ or $f(\complement U) \subset f(f^{-1}(\complement A)) \subset \complement A$. Since f is a-fuzzy closed and $\complement U$ is fuzzy closed in X. $f(\complement U) \subset Int(\complement A) = \complement(Cl(A))$. Thus $\complement U \subset f^{-1}(\complement Cl(A)) = \complement f^{-1}(Cl(A))$ and hence $f^{-1}(Cl(A)) \subset U$. Since f is g-fuzzy continuous and $Cl(A)$ is fuzzy closed in Y, $f^{-1}(Cl(A))$ is g-fuzzy closed in X. Thus $Cl(f^{-1}(A)) \subset Cl(f^{-1}(Cl(A))) \subset U$. Hence $f^{-1}(A)$ is g-fuzzy closed in X.

A similar argument shows that inverse image of g-open fuzzy sets are g-fuzzy open. ■

Theorem 3.2. *If $f : X \to Y$ is a-fuzzy continuous and fuzzy closed, then $f(A)$ is g-fuzzy closed in Y whenever A is g-fuzzy closed in X.*

Proof. Let A be any g-closed fuzzy set in X and let V be any open fuzzy set in Y such that $f(A) \subset U$. Then $A \subset f^{-1}(V)$. Since f is g-fuzzy continuous and A is g-fuzzy closed in X, $Cl(A) \subset f^{-1}(V)$. Thus $f(Cl(A)) \subset V$. Since f

s fuzzy closed, $f(Cl(A))$ is fuzzy closed in Y. So $Cl(f(A)) \subset Cl(f(Cl(A))) = f(Cl(A)) \subset V$, and hence $Cl(f(A)) \subset V$. Therefore $f(A)$ is g-fuzzy closed. ■

4. Some Properties of a-Fuzzy Closed and a-Fuzzy Continuous Mapping.

In this section we characterize $T_{\frac{1}{2}}$-spaces using a-fuzzy closed and a-fuzzy continuous mappings. Also we obtain sufficient conditions for a mapping to a-fuzzy closed or a-fuzzy continuous. Finally we investigate some of the properties of these mappings involving composition.

Definition 4.1. *A fuzzy topological space X is called a* $T_{\frac{1}{2}}$**-space** *if each g-closed fuzzy set in X is fuzzy closed.*

Theorem 4.2. *Let X be a fuzzy topological space. Then X is a $T_{\frac{1}{2}}$-space if and only if for each fuzzy topological space Y and each mapping $f : X \to Y$, f is a-fuzzy continuous.*

Proof. ($\Rightarrow$): Suppose X is a $T_{\frac{1}{2}}$-space, let V be any open fuzzy set in Y, and let A be a g-closed fuzzy set in X such that $A \subset f^{-1}(V)$. Then by hypothesis, $A = Cl(A)$. Thus $Cl(A) \subset f^{-1}(V)$. Hence f is a-fuzzy continuous.

($\Leftarrow$): Suppose the necessary condition holds. Let A be a (non-empty) g-closed fuzzy set in X and let Y be the set X with the fuzzy topology $\mathcal{U} = \{\underline{0}, \underline{1}, A\}$. Let $f : X \to Y$ be the identity mapping. Then clearly f is a-fuzzy continuous. Since A is g-fuzzy closed in X and fuzzy open in Y, and $A \subset f^{-1}(A)$, $Cl_X(A) \subset f^{-1}(A) = A$. So A is fuzzy closed in X. Therefore X is $T_{\frac{1}{2}}$. ■

The following result can be proved by the similar argument in Theorem 4.2

Theorem 4.3. *A fuzzy topological space X is $T_{\frac{1}{2}}$ if and only if for each fuzzy topological space Y and each mapping $f : X \to Y$, f is a-fuzzy closed.*

From Definition 2.1, we can easily show the following two results:

Theorem 4.4. *Let $f : X \to Y$ be a mapping for which $f(F)$ is fuzzy open in Y for each closed fuzzy set F in X. Then f is a-fuzzy closed.*

Theorem 4.5. *Let $f : X \to Y$ be a mapping for which $f^{-1}(V)$ is fuzzy closed in X for each open fuzzy set V in Y. Then f is a-fuzzy continuous.*

Since the identity mapping on any fuzzy topological space is both a-fuzzy continuous and a-fuzzy closed, it is clear that the converse of Theorem 4.4 and 4.5 do not hold.

Compositions of a-fuzzy continuous (or a-fuzzy closed) mappings are not in general a-fuzzy continuous (or a-fuzzy closed). However the following results hold:

Theorem 4.6. *If $f : X \to Y$ is fuzzy closed and $g : Y \to Z$ is a-fuzzy closed, then $g \circ f : X \to Z$ is a-fuzzy closed.*

Proof. Let F be any closed fuzzy set in X and let A be a g-open fuzzy set in Z such that $g \circ f(F) \subset A$. Since f is fuzzy closed, $f(F)$ is fuzzy closed in Y. Since g is a-fuzzy closed, $g(f(F)) \subset Int(A)$. Hence $g \circ f$ is a-fuzzy closed. ■

Theorem 4.7. *If $f : X \to Y$ is a-fuzzy closed and $g : Y \to Z$ is fuzzy open and inversely preserves g-open fuzzy sets, then $g \circ f : X \to Z$ is a-fuzzy closed.*

Proof. Let F be a closed fuzzy set in X and let A be a g-open fuzzy set in Z such that $g \circ f(F) \subset A$. Then $f(F) \subset g^{-1}(A)$. Since $g^{-1}(A)$ is g-fuzzy open in Y and f is a-fuzzy closed, $f(F) \subset Int(g^{-1}(A))$. Then $g \circ f(F) = g(f(F)) \subset g(Int(g^{-1}(A))) \subset Int(g(g^{-1}(A))) \subset Int(A)$. Hence $g \circ f$ is a-fuzzy closed. ■

Theorem 4.8. *If $f : X \to Y$ is a-fuzzy continuous and $g : Y \to Z$ is fuzzy continuous, then $g \circ f : X \to Z$ is a-fuzzy continuous.*

Proof. Let A be any g-closed fuzzy set in X and let V be an open fuzzy set in Z such that $A \subset (g \circ f)^{-1}(V)$. Since g is fuzzy continuous, $g^{-1}(V)$ is fuzzy open in Y. Since f is a-fuzzy continuous, $Cl(A) \subset f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Hence $g \circ f$ is a-fuzzy continuous. ■

Corollary 4.9. *Let $f_\alpha : X \to Y_\alpha$ be a mapping for each $\alpha \in J$ and let $f : X \to \Pi_{\alpha \in J} Y_\alpha$ be the product mapping given by $f(x) = (f_\alpha(x))$. If f is a-fuzzy continuous, then f_α is a-fuzzy continuous for each $\alpha \in J$.*

References.

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[2] K.Hur, A Note on g-closed fuzzy sets and g-fuzzy continuities.