A Note on g-Closed Fuzzy Sets and g-Fuzzy Continuities

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0.Introduction.

We introduce the concepts of generalized closed fuzzy set(breifly g-closed fuzzy set) and generalized fuzzy continuity(briefly g-fuzzy continuity), and investigate their some properties. When A is a fuzzy set in a fuzzy topological space, we denote the closure of A, the interior of A and the complement of A as cl(A), Int(A) and CA, respectively.

1.Preliminiaries.

Let X be a set and let $I = [0,1]$. Then a mapping $A : X \to I$ is called a **fuzzy set** in X [5]. We denote the family of all the fuzzy sets in X as I^X. In particular, we will denote the empty fuzzy set and the whole fuzzy set in X as $\underline{0}$ and $\underline{1}$, respectively, where $\underline{0}$ and $\underline{1}$ denotes the constant mapping from X into I with the values 0 and 1, respectively.

We begin with some preliminary definitions and results.

Definition 1.1 [1]. *A* **fuzzy topology** *on a set X is a family τ of fuzzy sets in X which satisfies the following conditions:*

(a) $\underline{0}, \underline{1} \in \tau$,

(b) If $A, B \in \tau$, then $A \cap B \in \tau$,

(c) If $A_\alpha \in \tau$ for each $\alpha \in J$, then $\cup_{\alpha \in J} A_\alpha \in \tau$.

The pair (X, τ) is called a **fuzzy topological space** (or simply fts.) Each member of τ is called a τ-**open** fuzzy set(or simply open fuzzy set) in X. A fuzzy set is τ-**closed** if and only if its complement is τ-open.

Definition 1.2[1]. *Let X and Y be sets, let $f : X \to Y$ be a mapping, and let $A \in I^X$ and $B \in I^Y$. Then:*

(1) The **inverse image** *of B, written as $f^{-1}(B)$, is a fuzzy set in X defined by*

$$[f^{-1}(B)](x) = B(f(x)) = (B \circ f)(x) \;\; for \;\; each x \in X$$

(2) The **image** *of A, written as $f(A)$, is a fuzzy set in Y defined by*

$$[f(A)](y) = \begin{cases} sup_{x \in f^{-1}(y)} A(x), & if \;\; f^{-1}(y) \neq \emptyset \\ 0, & otherwise. \end{cases}$$

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for each $y \in Y$, where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Theorem A[1]. *Let f be a mapping from a set X to a set Y, Then:*

(a) If $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in I^Y$.

(b) If $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$, where A_1, $A_2 \in I^X$.

(c) $f(f^{-1}(B)) \subset B$ for each $B \in I^Y$.

(d) $A \subset f^{-1}(f(A))$ for each $A \in I^X$.

(e) Let g be a mapping from a set Y to a set Z. Then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ for each $C \in I^Z$, where $g \circ f$ is the composition of f and g.

Definition 1.3 [4]. *Let (X, τ) and (Y, u) be fuzzy topological spaces. Then a mapping $f : X \to Y$ is said to be* **fuzzy continuous***(or simply F-continuous) if the inverse image of each u-open fuzzy set is τ-open.*

Theorem B[4]. *Let(X, τ) be the product fuzzy topological space of the family $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in J}$ of fuzzy topological spaces. Then:*

(a) For each $\alpha \in J$, π_α is fuzzy continuous.

(b) τ is the smallest fuzzy topology on X such that (a) is true.

(c) Let (Y, u) be a fuzzy topological space and let $f : Y \to X$ Then f is fuzzy continuous if and only if $\pi_\alpha \circ f$ is fuzzy continuous for each $\alpha \in J$, where π_α is the projection of X onto X_α.

Definition 1.4 [3]. *A fuzzy set A in a set X is called a* **fuzzy point** *of X with the value $\lambda (0 < \lambda \leq 1)$ and the support $x \in X$ if for each $y \in X$,*

$$A(y) = \begin{cases} \lambda, & \text{if} \quad y = x \\ 0, & \text{otherwise.} \end{cases}$$

We denote this fuzzy point A by x_λ.

Definition 1.5 [3]. *A fuzzy point x_λ is said to be* **contained in** *a fuzzy set A or to* **belong to** *A, denoted by $x_\lambda \in A$, if $\lambda \leq A(x)$.*

Definition 1.6[3]. *Let A and B be fuzzy sets in a set X. Then A is said to be* **quasi-coincident** *with B, denoted by AqB, if there exists $x \in X$ such that $A(x) > (CB)(x)$ or $A(x) + B(x) > 1$. If this is true, we also say that A and B are quasi-coincident(with each other) at x.*

Definition 1.7[3]. *A fuzzy point x_λ is said to be* **quasi-coincident** *with A, denoted by $x_\lambda q A$, if $\lambda > (CA)(x)$ or $\lambda + A(x) > 1$.*

Definition 1.8 [3]. *A fuzzy set A in a fuzzy topological space (X, τ) is called a* **Q-neighborhood** *of x_λ, if there exists a $B \in \tau$ such that $x_\lambda q B \subset A$.*

Theorem C[3]. *$A \subset B$ if and only if A and CB are not quasi-coincident. In particular, $x_\lambda \in A$ if and only if x_λ is not quasi-coincident with CA.*

Theorem D[3]. *A fuzzy point $x_\lambda \in Cl(A)$ if and only if each Q-neighborhood of x_λ is quasi-coincident with A.*

2. g-Closed Fuzzy Sets.

In this section we will introduce the concepts of g-closed fuzzy set and the g-closure, and obtain two results.

Definition 2.1. *Let X be a fuzzy topological space and let $A \in I^X$. Then:*

(1) A is said to be **g-fuzzy closed** *in X if $cl(A) \subset G$ whenever $A \subset G$ and G is fuzzy open in X.*

(2) A is said to be **g-fuzzy open** *in X if CA is g-fuzzy closed in X.*

Example 2.2. *Let $X = \{a, b, c\}$ and let $\tau = \{\underline{0}, O_1, O_2, O_3, O_4, 1\}$, where*

$$O_1 = \{(a, 0.5), (b, 0.6), (c, 0.8)\}, \;\; O_2 = \{(a, 0.4), (b, 0.7), (c, 0.2)\}$$
$$O_3 = \{(a, 0.4), (b, 0.6), (c, 0.2)\}, \;\; O_4 = \{(a, 0.5), (b, 0.7), (c, 0.8)\}$$

Let $A = \{(a, 0.5), (b, 0.2), (c, 0.1)\}$. Then (X, τ) is a fuzzy topological space and A is a g-fuzzy closed in X.

The following result gives a useful characterization of g-fuzzy openness:

Theorem 2.3. *Let A be a fuzzy set in a fuzzy topological space X. Then A is g-fuzzy open in X if and only if $F \subset Int(A)$ whenever F is fuzzy closed in X and $F \subset A$.*

Proof. $(\Rightarrow)$: Suppose A is g-fuzzy open in X. Let F be any closed fuzzy set in X such that $F \subset A$. Then CF is Fuzzy open in X such that $CA \subset CF$. By hypothesis, since CA is g-fuzzy closed in X, $cl(CA) \subset CF$. Let $\mathcal{F} = \{G \in I^X | G \text{ is closed in } X \text{ and } CA \subset G\}$ and let $x \in X$. Then there is a $G \in \mathcal{F}$ such that $G(x) \leq (CF)(x) = 1 - F(x)$. Since $G \in F$, $CA \subset G$. Thus $1 - A(x) + (CA)(x) \leq G(x)$. So $F(x) \leq (CG)(x) \leq A(x)$ and CG is fuzzy open in X. Hence $F(x) \leq (\cup_{G \in \mathcal{F}} CG)(x) = [Int(A)](x)$. Therefore $F \subset Int(A)$.

$(\Leftarrow)$: Suppose the necessary condition holds. Let U be any open fuzzy set in X such that $CA \subset U$. Then CU is fuzzy closed in X and $CU \subset A$. By hypothesis, $CU \subset Int(A)$. Let $\mathcal{O} = \{O \in I^X | O \text{ is open in } X \text{ and } O \subset A\}$ and let $x \in X$. Then there is an $O \in \mathcal{O}$ such that $1 - U(x) = (CU)(x) \leq O(x)$. Since $O \in \mathcal{O}$, $A \subset O$. Thus $O(x) \leq A(x)$. So $1 - A(x) \leq 1 - O(x) \leq U(x)$ and hence $(CA)(x) \leq (CO)(x) \leq U(x)$. Since CO is fuzzy closed in X, $(\cap_{O \in \mathcal{O}} CO)(x) \leq U(x)$, that is, $[cl(CA)](x) \leq U(x)$. Thus $cl(CA) \subset U$ and hence CA is g-fuzzy closed in X. Therefore A is g-fuzzy open in X.

Definition 2.4. *Let A be a fuzzy set in a fuzzy topological space X. Then the intersection of all* g-closed *fuzzy sets containing A is called the* **g-closure** *of A and is denoted by $cl^*(A)$. Hence*

$$cl^*(A) = \cap\{F \in I^X | F \text{ is g-fuzzy closed in } X \text{ and } A \subset F\}$$

for each $A \in I^X$.

It is clear that $A \subset cl^*(A) \subset cl(A)$. We shall assume that $cl^*(A)$ is g-fuzzy closed in X. It is easily proved that A is g-fuzzy closed in X if and only if $cl^*(A)$ is g-fuzzy closed in X.

Definition 2.5. *Let x_λ be a fuzzy point of X and let $N \in I^X$. Then N is a called* **g-Q-neighborhood** *of x_λ in X if there is a* g-*open fuzzy set O in X such that $x_\lambda q O \subset N$.*

Theorem 2.6. *Let$A \in I^X$. Then $x_\lambda \in cl^*(A)$ if and only if for each* g-*Q-neighborhood N_{x_λ} of x_λ in X, $N_{x_\lambda} q A$.*

Proof. ($\Rightarrow$): Suppose $x_\lambda \in cl^*(A)$. Assume that necessary condition does not hold. Then there is a g-Q-neighborhood N of x_λ in X such that $N \not{q} A$. Since N is a g-Q-neighborhood of x_λ, by Definition 2.5, there is a g-open fuzzy set O in X such that $x_\lambda q O \subset N$. Thus $\lambda + O(x) > 1$ and $O(x) \leq N(x)$. Since $N \not{q} A$, $N(x) + A(x) \leq 1$. Thus $O(x) + A(x) \leq N(x) + A(x)$. So $O(x) + A(x) \leq 1$ and hence $O \not{q} A$. Therefore $A \subset CO$. By Definition 2.1, CO is g-closed in X. Thus $cl^*(A) \subset CO$. Since $\lambda + O(x) > 1$, $x_\lambda \notin CO$. So $x_\lambda \notin cl^*(A)$. This is contrary to the hypothesis.

($\Leftarrow$): Suppose the necessary condition holds. Assume that $x_\lambda \notin cl^*(A)$. By Definition 2.4, there exists a g-closed set F of X such that $A \subset F$ and $x_\lambda \notin F$. Thus $x_\lambda q CF$ and CF is g-open fuzzy set in X. Thus CF is a g-Q-neighborhood of x_λ in X. But $CF \not{q} A$. This contrary to the hypothesis.

3. g-Fuzzy Continuities.

In this section we will introduce the concept of g-fuzzy continuity and obtain some properties of g-fuzzy continuities.

Definition 3.1. *Let X and Y be fuzzy topological spaces. Then a mapping $f : X \to Y$ is said to be* **g -fuzzy continuous** *if the inverse image of every closed fuzzy set in Y is* g-*fuzzy closed in X.*

It is clear that a mapping $f : X \to Y$ is g -fuzzy continuous if and only if the inverse image of every open fuzzy set in Y is g-fuzzy open in X.

Example 3.2. *Let (X, τ) be the fuzzy topological space in Example 2.2. Let $Y = \{x, y, z, w\}$ and let $u = \{\underline{0}, O, \underline{1}\}$, where $O = \{(x, 0.7), (y, 0.5), (z, 0.8), (w, 0.2)\}$. We define a mapping $f : X \to Y$ by $f(a) = f(b) = x$, $f(c) = w$. Then $f : (X, \tau) \to (Y, u)$ is* g-*fuzzy continuous.*

Theorem 3.3. *Let $\{X_\alpha | \alpha \in J\}$ be a family of fuzzy topological spaces and let X be a fuzzy topological space. If $f : X \to \Pi_{\alpha \in J} X_\alpha$ is a g-fuzzy continuous mapping, then $\Pi_\alpha \circ f : X \to X_\alpha$ is a g- fuzzy continuous for each $\alpha \in J$, where π_α is the projection of $\Pi_{\alpha \in J} X_\alpha$ onto X_α.*

Proof. For each $\alpha \in J$, let U_α be an arbitrary open fuzzy set in X_α. By Theorem B (a), since π_α is fuzzy continuous, $\pi_\alpha^{-1}(U_\alpha)$ is fuzzy open in $\Pi_{\alpha \in J} X_\alpha$. By hypothesis, and Definition 3.1, $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha)$ is g-fuzzy open in X. Therefore $\pi_\alpha \circ f$ is g-fuzzy continuous.

Theorem 3.4. *Let X and Y be fuzzy topological spaces, and let $f : X \to Y$ be g-fuzzy continuous and fuzzy closed. If G is g-fuzzy open(resp. g-fuzzy closed) in Y, then $f^{-1}(G)$ is g-fuzzy open(resp. g-fuzzy closed) in X.*

Proof. Let G be a g-open fuzzy set in Y. Let $F \subset f^{-1}(G)$, where F is fuzzy closed in X. Then $f(F) \subset G$ and $f(F)$ is fuzzy closed in Y because f is a fuzzy closed mapping. Since G is g-fuzzy open in Y, by Theorem 2.3, $f(F) \subset Int(G)$. Thus $F \subset f^{-1}(Int(G))$. Since f is g-fuzzy continuous and $Int(G)$ is fuzzy open in Y, by Theorem 2.3, $F \subset Int[f^{-1}(Int(G))] \subset Int(f^{-1}(G))$. Hence $f^{-1}(G)$ is g-fuzzy open in X. By taking complements, we can show that if G is g-fuzzy closed in Y, then $f^{-1}(G)$ is g-fuzzy closed in X.

We obtain the following result immediately from Theorem 3.4.

Corollary 3.5. *Let X, Y and Z be fuzzy topological spaces. If $f : X \to Y$ is a fuzzy closed and g-fuzzy continuous mapping, and $g : Y \to Z$ is a g-fuzzy continuous mapping, then $g \circ f : X \to Z$ is g-fuzzy continuous.*

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