# 퍼지 바운드 사이의 실사상의 적분의 유사평균치정리

# LIKELY MEAN VALUE THEOREM OF INTEGRALS OF REAL MAPPING BETWEEN FUZZY BOUNDS

YOUNG-CHEL KWUN, DONG-GUN PARK AND MI-JUNG SON

ABSTRACT

We study likely mean value theorem with respect to integral of real mapping between fuzzy bound. This is the main purpose of this paper, which investigates ideas in Dubois & Prade ([2,3,4]).

## DEFINITIONS AND MAIN RESULTS

A fuzzy domain $D$ of the real line $R$ is assumed to be delimited by two fuzzy bounds $\bar{a}$ and $\bar{b}$ in the following sence:

(i) $\bar{a}$ and $\bar{b}$ are fuzzy sets on $R$, whose membership functions are $\mu_{\bar{a}}$ and $\mu_{\bar{b}}$, from $R$ to $[0,1]$,

(ii) For all $x \in R$, $\mu_{\bar{a}}(x)$(resp. $\mu_{\bar{b}}(x)$) evaluates to what extent $x$ can be cosidered as a greatest lower bound (resp. least upper bound) of $D$,

(iii) $\bar{a}$ and $\bar{b}$ are normalized, ie., there exists $a, b \in R$ such that $\mu_{\bar{a}}(a) = 1 = \mu_{\bar{b}}(b)$,

(iv) $\bar{a}$ and $\bar{b}$ are convex fuzzy sets, i.e., $\forall\alpha \in (0.1]$ their $\alpha -$ cuts $\bar{a}_\alpha$ and $\bar{b}_\alpha$ are intervals.

$D$ is denoted $(\bar{a}, \bar{b})$ : $\bar{a}$ and $\bar{b}$ are assumed ordered in the sense that

$$\underline{a}_0 = \inf S(\bar{a}) \leq \sup S(\bar{b}) = \bar{b}_0$$

where $S(\bar{a}) = \{x|\mu_{\bar{a}}(x) > 0\}$ is support of $\bar{a}$ ( See Dubois & Parde [3]).

**Definition 1([3])** Let $f$ be a real-valued real mapping, supposedly integrable on the interval $I = [\inf S(\bar{a}), \sup S(\bar{b})]$; then the integral of $f$ over the domain delimited by the fuzzy bounds $\bar{a}$ and $\bar{b}$, denoted $\int_D f$, is defined according to the extension principle by

$$\forall z \in R,\ \mu_{\int_D f}(z) = \sup_{x,y \in I} \min(\mu_{\bar{a}}(x),\ \mu_{\bar{b}}(y))$$

under the constraint $z = \int_x^y f$, where $\int_x^y f$ is short for $\int_x^y f(s)ds$. $\int_D f$ will also be denoted $\int_{\bar{a}}^{\bar{b}} f$.

**Definition 2([4])** Fuzzy point is convex subset of real line $R$ and its membership function is defined by

$$\forall x,\ \forall y > x,\ \forall z \in [x,y],\ \mu_c(z) \geq \min(\mu_c(x),\ \mu_c(y)).$$

**Theorem 1.** *Let $\bar{a}$ and $\bar{b}$ are bounded normal fuzzy domain on $R$ and $f$ be a real valued mapping supposedly integrable on the interval* [inf $S(\bar{a})$, sup $S(\bar{b})$] *then there exists fuzzy point $\bar{c}$ satisfying*

$$\int_{\bar{a}}^{\bar{b}} f(s)ds \subseteq f(\bar{c})(\bar{b} \ominus \bar{a})$$

*where* $S(\bar{c}) \subset [\inf S(\bar{a}), \sup S(\bar{b})]$.

*Proof.* By definition 1,

$$\mu_{\int_{\bar{a}}^{\bar{b}} f}(z) = \sup_{\int_w^u f = z} \min\{\mu_{\bar{a}}(w), \mu_{\bar{b}}(u)\}.$$

Since $\int_w^u f$ is Riemann integral, by ordinary mean value theorem, there exist $t\,(w < t < u)$ satisfy

$$\int_w^u f(s)ds = f(t)(u-w).$$

Thus

$$\mu_{\int_{\bar{a}}^{\bar{b}} f}(z) = \sup_{xy=z} \min\{ \sup_{\substack{t:x=f(t)\\ w<t<u}} \min\{\mu_{\bar{a}}(w), \mu_{\bar{b}}(u)\}, \sup_{u-w=y} \min\{\mu_{\bar{a}}(w), \mu_{\bar{b}}(u)\}\}$$

We define membership function of $\bar{c}$ such that

$$\mu_{\bar{c}}(w) = \begin{cases} \mu_{\bar{a}}(w), & w \in S(\bar{a}) \\ \mu_{\bar{b}}(w), & w \in S(\bar{b}) \end{cases}$$

Since $w \in S(\bar{a})$ and $u \in S(\bar{b})$,

$$\mu_{\int_{\bar{a}}^{\bar{b}} f}(z) = \sup_{xy=z} \min\{ \sup_{\substack{t:x=f(t)\\ w<t<u}} \min\{\mu_{\bar{c}}(w), \mu_{\bar{c}}(u)\}, \sup_{u-w=y} \min\{\mu_{\bar{a}}(w), \mu_{\bar{b}}(u)\}\}.$$

By definition of fuzzy point,

$$\min\{\mu_{\bar{c}}(w), \mu_{\bar{c}}(u)\} \le \mu_{\bar{c}}(t), \quad w < t < u,$$

$$\mu_{\int_{\bar{a}}^{\bar{b}} f}(z) \le \sup_{xy=z} \min\{ \sup_{\substack{t:x=f(t)\\ w<t<u}} \mu_{\bar{c}}(t), \sup_{u-w=y} \min\{\mu_{\bar{a}}(w), \mu_{\bar{b}}(u)\}\}.$$

By extension principal,

$$\sup_{u-w=y} \min\{\mu_{\bar{a}}(w), \mu_{\bar{b}}(u)\} = \mu_{\bar{b}\ominus\bar{a}}(y)$$

and

$$\sup_{\substack{t:x=f(t)\\ w<t<u}} \mu_{\bar{c}}(t) = \mu_{f(\bar{c})}(x).$$

Hence

$$\mu_{\int_{\bar{a}}^{\bar{b}} f}(z) \le \sup_{xy=z} \min\{\mu_{f(\bar{c})}(x), \mu_{\bar{b}\ominus\bar{a}}(y)\}.$$

Using extension principal,

$$\mu_{\int_{\bar{a}}^{\bar{b}} f} \le \mu_{f(\bar{c})(\bar{b}\ominus\bar{a})}(z).$$

**Corollary.** *Under assumption of theorem 1, we give membership function of* $\bar{a}$ *and* $\mu_{\bar{b}}(y) = 1$ *then we define membership function of* $\bar{b}$. *Furthermore, in this case also satisfy theorem 1.*

*Proof.* It suffices to define membership function of $\bar{b}$. Let $x \in S(\bar{a})$ and $\int_x^y f(s)ds = z$ then there exists $S(\bar{b}) = \{y \mid \int_x^y f(s)ds = z\}$. Put $\mu_{\bar{b}}(k) = 1$. Define

$$\mu_{\bar{b}}(y) = \begin{cases} \dfrac{y - \inf S(\bar{b})}{b - \inf S(\bar{b})}, & y < k \\ \dfrac{\sup S(\bar{b}) - y}{\sup S(\bar{b}) - b}, & y > k. \end{cases}$$

## References


1. R. J. Aumann, *Integrals of set valued mappings*, J. Math. Anal. Appl. 12 (1965), 1–12.
2. D. Dubois and H. Prade, *Towards fuzzy differential part 1 : Integration of fuzzy mappings*, Fuzzy Sets and System 8 (1982), 1–17.
3. ———, *Towards fuzzy differential part 2 : Integration on fuzzy intervals*, Fuzzy Sets and System 8 (1982), 105–6.
4. ———, *Towards fuzzy differential part 3 : Differentiation*, Fuzzy Sets and System 8 (1982), 225–33.
5. M. Mizumoto and K. Tanaka, *Some properties of fuzzy numbers*, *Advanced in fuzzy set theory and application*, North-Holland, 1979, pp. 153–64.
6. H. Nguyen, *A note on the extension principle for fuzzy sets*, J. Math. Anal. Appl. 64 (1978), 369–80.
7. L. A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and System 1 (1978), 3–28.
8. H.-J. Jimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Publ., 1991.



Department of Mathematics, Dong-A University, Pusan 604-714, Korea