퍼지 정칙 수렴구조

FUZZY REGULAR CONVERGENCE STRUCTURES

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ABSTRACT

We define a fuzzy regular modification and fuzzy regular series. And we investigate some properties of fuzzy regular modification with respect to fuzzy initial convergence structure

1. INTRODUCTION AND PRELIMINARIES

We introduced in [7] the concept of fuzzy convergence structure using prefilter which was an extension of fuzzy topology, and c-neighborhood prefilter as a generalization of a q-neighborhood system in fuzzy topological space. And in [8] and [9], we defined the fuzzy initial convergence and separation axioms, investigated their some properties.

Our main purpose in this paper is to define fuzzy R-Hausdorff convergence structure, fuzzy regular modification and fuzzy regular series, and to investigate their some properties.

For a nonempty set X and unit interval $I = [0,1]$, a *fuzzy set* in X is an element of the set of all functions from X to I. The *support* of a fuzzy set A in X, Supp(A), is the crisp set of all $x \in X$ such that $A(x) > 0$. A *fuzzy point* p in X is a fuzzy set in X defined by $p(x_p) = \lambda_p (0 < \lambda_p \preceq 1)$ and $p(x) = 0$ for $x \neq x_p$, in this case, support of p is $\{x_p\}$, briefly x_p, and λ_p is called *value* of p. Fuzzy point p is called *fuzzy element* of fuzzy set A, $p \in A$ as notation, if $\lambda_p \preceq A(x_p)$. And p is said to be *quasi-coincident* with A, denoted by pqA, if $\lambda_p + A(x_p) > 1$. $F(X)$ denotes the set of all prefilters on X and $P(X)$ the set of all fuzzy sets on X.

Let $\mathcal{F}, \mathcal{G} \in F(X)$, $\mathcal{F}$ is said to be *finer* than $\mathcal{G}$ if $\mathcal{G} \leq \mathcal{F}$. Any two prefilter $\mathcal{F}$ and $\mathcal{G}$ are said to be *disjoint* if there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = 0_X$. $\mathcal{F} \in F(X)$ is called *ultra prefilter* if there is no other prefilter finer than $\mathcal{F}$ (i.e., it is maximal for the inclusion relation among prefilters) ([3]).

Let f be a function from X to Y, A be a fuzzy set in X and B be a fuzzy set in Y. Then $f(A)$ is a fuzzy set in Y with membership function

$$f(A)(y) = \begin{cases} \sup\{A(x) | x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

for any $y \in Y$, and $f^{-1}(B)$ is a fuzzy set in X whose membership function is defined by $f^{-1}(B)(x) = B(f(x))$ for any $x \in X$.

Lemma 1.1 ([3]). *If $f : X \to Y$ is a function and $\mathcal{F}$ is an ultra prefilter on X, then $f(\mathcal{F})$ is also an ultra prefilter on Y.*

Let $X = \prod_{i\in\Lambda} A_i$ be the cartesian product of X_i's and for each $i \in \Lambda$ A_i be a fuzzy set in X_i. Then $\prod_{i\in\Lambda} A_i$ is a fuzzy set in X is defined by

$$\prod_{i\in\Lambda} A_i(x) = \inf\{A_i(x_i) | i \in \Lambda\}$$

for any $x = (x_i)_{i\in\Lambda} \in X$.

A *fuzzy convergence structure* on X is a function c from $F(X)$ into $P(X)$ satisfying the following conditions :

(1) For each fuzzy point p in X, $p \in c(\dot{p})$.
(2) For $\mathcal{F}, \mathcal{G} \in F(X)$, if $\mathcal{F} \leq \mathcal{G}$, then $c(\mathcal{F}) \leq c(\mathcal{G})$.
(3) If $p \in c(\mathcal{F})$, then $p \in c(\mathcal{F} \cap \dot{p})$.

The pair (X, c) is called *fuzzy convergence space*. If $p \in c(\mathcal{F})$, we say that $\mathcal{F}$ *c-converges* to p. The prefilter $\mathcal{V}_c(p)$ obtained by intersecting all prefilters which c-converge to p is called the *c-neighborhood prefilter* at p. If $\mathcal{V}_c(p)$ c-converges to p for each fuzzy point p in X, then c is called *fuzzy pretopological structure*, and (X, c) *fuzzy pretopological space* ([7]).

Theorem 1.2 ([7]). *Let $\mathcal{C}(X)$ be the set of all fuzzy convergence structures on X and $\preceq$ a relation on $\mathcal{C}(X)$ defined by for each $c_1, c_2 \in \mathcal{C}(X)$,*

$$c_1 \preceq c_2 \text{ if and only if } c_2(\mathcal{F}) \leq c_1(\mathcal{F}) \text{ for any } \mathcal{F} \in F(X).$$

Then $\preceq$ is a partially order on $\mathcal{C}(X)$.

If $c_1 \preceq c_2$, we say that c_2 is *finer* than c_1 and c_1 is *coarser* than c_2.

The partially ordered set $\mathcal{C}(X)$ is complete lattice whose greatest and least elements are the crip discrete and indiscrete topologies respectively. Thus the following theorem is obtained.

Theorem 1.3. *For any $\mathcal{S} \subset \mathcal{C}(X)$, $\sup\mathcal{S}$ and $\inf\mathcal{S}$ are given by the following statements.*

$$(\sup\mathcal{S})(\mathcal{F}) = \inf\{c(\mathcal{F}) | c \in \mathcal{S}\}, \quad (\inf\mathcal{S})(\mathcal{F}) = \sup\{c(\mathcal{F}) | c \in \mathcal{S}\}.$$

Let (X, c) be a fuzzy convergence space and Γ_c be fuzzy closure operator in X, defined by

$$\Gamma_c(A) = \{p \mid p \in c(\mathcal{F}) \text{ for some prefilter } \mathcal{F} \text{ on } X \text{ with } A \in \mathcal{F}\}$$

for each $A \in P(X)$. Γ_c has all properties of fuzzy topological closure operator except idempotency. A fuzzy set A in X is said to be *dense* if $\Gamma_c(A) = 1_X$.

Theorem 1.4 ([7]). *Let c_1, c_2 be fuzzy convergence structures on X and A be a fuzzy set in X. If $c_1 \preceq c_2$, then $\Gamma_{c_2}(A) \leq \Gamma_{c_1}(A)$.*

Let $\Gamma_c^0(A) = A$. If α is an ordinal number and $\alpha - 1$ exists, then $\Gamma_c^\alpha(A)$ is defined by $\Gamma_c(\Gamma_c^{\alpha-1}(a))$, IF α is a limit ordinal, then $\Gamma_c^\alpha(A)$ is defined by the $\sup\{\Gamma_c^\beta(A)|\beta < \alpha\}$.

Let f be a function from a fuzzy convergence space (X, c_X) onto a fuzzy convergence space (Y, c_Y). f is said to be ***fuzzy continuous*** at a fuzzy point p in X if the prefilter $f(\mathcal{F})$ c_Y-converges to $f(p)$ for every prefilter $\mathcal{F}$ c_X-converging to p. If f is fuzzy continuous at any fuzzy point p in X, then f is said to be ***fuzzy continuous***.

Theorem 1.5 ([7]). *If f is fuzzy continuous and A is a fuzzy set in X, then $f(\Gamma_{c_X}(A)) \leq \Gamma_{c_Y}(f(A))$.*

Theorem 1.6. *If f is fuzzy continuous and A is a fuzzy set in X, then $f(\Gamma^n_{c_X}(A)) \leq \Gamma^n_{c_Y}(f(A))$ for each natural number n.*

Let X be a set, (X_i, c_i) be a fuzzy convergence space for each $i \in \Lambda$, $f_i : X \to (X_i, c_i)$ be a surjection. c is a map from $F(X)$ to $P(X)$ satisfying the following condition :

for any fuzzy point p in X and $\mathcal{F} \in F(X)$, $p \in c(\mathcal{F})$ if and only if $f_i(\mathcal{F})$ c_i-converges to $f_i(p)$ for each $i \in \Lambda$.

Then by [8], c is a fuzzy convergence structure on X that is called the ***fuzzy initial convergence structure*** induced by the family $\{f_i|i \in \Lambda\}$

Hereafter (X, c) $((\prod_{i\in\Lambda} X_i, c') = \prod_{i\in\Lambda}(X_i, c_i))$ means fuzzy initial convergence space induced by $\{f_i \mid f_i : X \longrightarrow (X_i, c_i)\,;\text{onto},\ i \in \Lambda\}$ (resp. $\{(P_i \mid P_i : \prod_{i\in\Lambda} X_i \to (X_i, c_i);\text{projection},\ i \in \Lambda\}$). And c' is called ***fuzzy product convergence structure of*** $\{(X_i, c_i) \mid i \in \Lambda\}$.

Theorem 1.7 ([8]). *Let c_i, d_i be fuzzy convergence structures in X_i for each $i \in \Lambda$. If $c_i \preceq d_i$ for each $i \in \Lambda$, then $c \preceq d$, where d is fuzzy initial convergence structure in X induced by $\{f_i|f_i X \to (X_i, d_i),\ i \in \Lambda\}$.*

A fuzzy convergence space (W, e) is called fuzzy ***Hausdorff***, if prefilter $\mathcal{F}$ e-converges to t and w, then $x_t = x_w$.

Theorem 1.8 ([9]). (1) *Let (X_i, c_i) be a fuzzy convergence space for each $i \in \Lambda$. If there exists at least one $j \in \Lambda$ such that f_j is injection and X_j is fuzzy Hausdorff, then (X, c) is fuzzy Hausdorff convergence space.*

(2) *(X_i, c_i) is fuzzy Hausdorff convergence space for each $i \in \Lambda$ if and only if $(\prod_{i\in\Lambda} X_i, c')$ is also fuzzy Hausdorff convergence space.*

Let e, o be two fuzzy convergence structures on a set W. (W, e) is said to be ***fuzzy o-regular*** if $\Gamma_o(\mathcal{F})$ e-converges to w whenever $\mathcal{F}$ e-converges to w, and fuzzy o-regular Hausdorff convergence space is called ***fuzzy o-T_3 convergence space***

Fuzzy convergence space (W, e) is said to be ***fuzzy regular*** if $\Gamma_e(\mathcal{F})$ e-converges to w whenever $\mathcal{F}$ e-converges w, and fuzzy regular Hausdorff convergence space is said to be ***fuzzy T_3 convergence space***.

Theorem 1.9 ([9]). *Let c_i, e_i be fuzzy convergence structures on X_i for each $i \in \Lambda$. If (X_i, c_i) is fuzzy e_i-regular convergence space for each $i \in \Lambda$, then the following statements are hold :*

(1) (X, c) is fuzzy e-regular, where e is the fuzzy initial convergence structure induced by the family $\{f_i \mid f_i : X \longrightarrow (X_i, e_i),\ i \in \Lambda\}$.

(2) $(\prod_{i\in\Lambda} X_i, c')$ is fuzzy e'-regular convergence space, where e' is fuzzy product convergence structure of e_i on $\prod_{i\in\Lambda} X_i$.

Furthermore, If (X_i, c_i) is fuzzy regular convergence space for each $i \in \Lambda$, then

(3) (X, c) is also fuzzy regular convergence space.

(4) $(\prod_{i\in\Lambda} X_i, c')$ is fuzzy regular convergence space.

2. Fuzzy regularities

Theorem 2.1. *Let c_i be a fuzzy e_i-regular convergence structure on X for each $i \in \Lambda$. Then $c = \sup\{c_i | i \in \Lambda\}$ is a fuzzy e-regular convergence structure on X, where $e = \sup\{e_i | i \in \Lambda\}$.*

By Theorem 2.1, the supremum $r(c)$ of all fuzzy regular convergence structures coarser than c on X is the finest fuzzy regular convergence structure on X coarser than c, is called the ***fuzzy regular modification*** of c. The idea of our notions is essentially due to the work of G.D. Richardson and D.C. Kent in [10].

A fuzzy convergence space (W, e) is said to be fuzzy *R-Hausdorff* if $(W, r(e))$ is fuzzy Hausdorff.

Theorem 2.2. *Let c^* be the fuzzy initial convergence structure on X induced by the family $\{f_i \mid f_i : X \longrightarrow (X_i, r(c_i)),\ i \in \Lambda\}$. Then $r(c) \succeq c^*$.*

Let c^* be fuzzy convergence structure defined in Theorem 2.2. Then by Theorem 1.8 and 2.2, the following corollaries are easily verified.

Corollary 2.3. $(\prod_{i\in\Lambda} X_i, r(c')) \succeq \prod_{i\in\Lambda}(X_i, r(c_i))$.

Corollary 2.4. *Let (X_i, c_i) be fuzzy R-Hausdorff convergence space for each $i \in \Lambda$. Then*

(1) If there exists at least one $j \in \Lambda$ such that f_j is injection, then (X, c^*) is fuzzy Hausdorff convergence space.

(2) (X, c) is fuzzy R-Hausdorff convergence space.

Corollary 2.5. *Let $X = \prod_{i\in\Lambda} X_i$. Then the following statements are equivalent.*

(1) (X_i, c_i) is fuzzy R-Hausdorff convergence space for each $i \in \Lambda$.

(2) (X, c^*) is fuzzy Hausdorff convergence space.

Furthermore, if (X_i, c_i) is fuzzy R-Hausdorff convergence space for each $i \in \Lambda$, then $(X, r(c'))$ is fuzzy Hausdorff.

Let (W, e) be a fuzzy convergence space. The set of all natural numbers (including 0) will be defined by N.

$$
\begin{aligned}
r_0(e) &= e\ ;\\
r_1(e)(\mathcal{F}) &= \{w \mid \text{there exist } n \in N \text{ and prefilter } \mathcal{G}\\
&\qquad\qquad \text{such that } w \in e(\mathcal{G}),\ \mathcal{F} \geq \Gamma^n_{r_1(e)}(\mathcal{G})\}\ ;\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ ,\\
r_\alpha(e)(\mathcal{F}) &= \{w \mid \text{there exist } n \in N \text{ and prefilter } \mathcal{G},\ \beta < \alpha\\
&\qquad\qquad \text{such that } w \in e(\mathcal{G}),\ \mathcal{F} \geq \Gamma^n_{r_\beta(e)}(\mathcal{G})\}.
\end{aligned}
$$

The family $\{W, r_\alpha(e))\}$ will be called the *fuzzy R-series* (or *fuzzy regularity series*) of (W, e).

Theorem 2.6. *If (X, c^*_α) is the fuzzy initial convergence space induced by the family $\{f_i \mid f_i : X \longrightarrow (X_i, r_\alpha(c_i)),\ i \in \Lambda\}$ for any ordinal number α. Then $r_\alpha(c) \succeq c^*_\alpha$.*

Thus, the following corollary is easily obtained.

Corollary 2.7. *If $(\prod_{i\in\Lambda} X_i, c')$ is the fuzzy product convergence space of family $\{(X_i, c_i) \mid i \in \Lambda\}$. Then, for any ordinal number α,*

$$\left(\prod_{i\in\Lambda} X_i, r_\alpha(c')\right) \succeq \prod_{i\in\Lambda}(X_i, r_\alpha(c_i)).$$

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