SOME CHARACTERIZATIONS OF FUZZY SUBGROUPS: VIA FUZZY p^*–SUBSETS AND FUZZY p^*–SUBGROUPS

JAE-GYEOM KIM AND HAN-DOO KIM

DEPARTMENT OF MATHEMATICS
KYUNGSUNG UNIVERSITY
PUSAN 608-736, KOREA
AND
DEPARTMENT OF MATHEMATICS
INJĖ UNIVERSITY
KIMHAE, KYŎNGNAM 621-749, KOREA

Abstract: The notion of A_{p^*} of a fuzzy subgroup A is introduced. Using the notion, we characterize fuzzy subgroups and show that every commutative fuzzy subgroup characterized as the intersection of its all minimal fuzzy p^*-subgroups.

1. Introduction. To grasp efficiently structures of fuzzy subgroups, the problem characterizing a fuzzy subgroup as the intersection of its all minimal fuzzy p-subgroups or the intersection of its all minimal fuzzy p^*-subgroups whose structures are quite more simple than the structure of the fuzzy subgroup has been studied [1,2,3,4]. In [1], fuzzy subgroups are characterized using the notion of A_p of a fuzzy subgroup A and conditions for a fuzzy subgroup to be written as the intersection of its all minimal fuzzy p-subgroups are given. However, the notion of A_p can be applied only to fuzzy subgroups in which all elements have finite fuzzy orders. In this paper, we introduce the notion of A_{p^*} of a fuzzy subgroup A that is free from the limit and an extension of the notion of A_p, characterize all fuzzy subgroups using the notion, and show that every commutative fuzzy subgroup characterized as the intersection of its all minimal fuzzy p^*-subgroups.

2. A_{p^*} of a fuzzy subgroup A. Let A be a fuzzy subgroup of a group G. If there exists a minimal fuzzy p-subgroup of G containing A, then it is unique [3] and we shall denoted it by $A_{(p)}$. And there exists a unique minimal fuzzy

This is a part of a forthcoming paper with the same title .

Typeset by $\mathcal{AMS}$-TEX

p^*-subgroup of G containing A [4] and we shall denote it by $A_{(p)^*}$. Note that a fuzzy p-subgroup always is a fuzzy p^*-subgroup.

To study which fuzzy subgroup can be written as the intersection of its all minimal fuzzy p-subgroups, the notion of A_p was introduced where A is a fuzzy subgroup of a group G such that $FO_A(x)$ is finite for all $x \in G$ in [1] as follows: For a given prime p, define a fuzzy subset A_p of G by $A_p(x) = A(x_{A_p})$ where $FO_A(x) = mp^t, (m,p) = 1, x = x'_{A_p} x_{A_p} = x_{A_p} x'_{A_p}, FO_A(x'_{A_p}) = m$, and $FO_A(x_{A_p}) = p^t$.

The notion of A_p can be applied only to fuzzy subgroups of groups whose every element has a finite fuzzy order. To overcome such limit, we now introduce the notion of A_{p^*}. While A_p corresponds to $A_{(p)}$, A_{p^*} corresponds to $A_{(p)^*}$. And the notion of A_{p^*} is a generalization of the notion of A_p (see Theorem 2.8).

DEFINITION 2.1. *Let A be a fuzzy subgroup of a group G. For a given prime p, define a fuzzy subset A_{p^*} of G by $A_{p^*}(x) = \sup\{A(x^n) | n \in \mathbb{N}, (n,p) = 1\}$.*

By Definition 2.1, it is clear that $A_{p^*} \supseteq A$. However $A_{p^*} \neq A$ and A_{p^*} is not a fuzzy subgroup in general.

PROPOSITION 2.2. *Let A be a fuzzy subgroup of a group G. For every $x \in G, \min\{n \in \mathbb{N} | A_{p^*}(x) < A_{p^*}(x^n)\}$ is a power of p, whenever this minimum exists.*

Motivating this proposition, we introduce the notion of a fuzzy p^*-subset as follows:

DEFINITION 2.3. *Let A be a fuzzy subset of a group G. For a given prime p, A is said to be a fuzzy p^*-subset of G if, for every $x \in G, \min\{n \in \mathbb{N} | A(x) < A(x^n)\}$ is a power of p, whenever this minimum exists.*

DEFINITION 2.4. *Let A be a fuzzy subset of a group G. For a given prime p, A is said to be a fuzzy p-subset of G if $FO_A(x)$ is a power of p for all $x \in G$.*

PROPOSITION 2.5. *Let f be a group homomorphism from G onto H. Then the following hold:*

(1) If A is a fuzzy p^*-subset [resp. p-subset] of G, then $f(A)$ is a fuzzy p^*-subset [resp. p-subset] of H, provided A is f-invariant, i.e., if $f(x) = f(y)$ implies $A(x) = A(y)$.

(2) If B is a fuzzy p^*-subset [resp. p-subset] of H, then $f^{-1}(B)$ is a fuzzy p^*-subset [resp. p-subset] of G.

COROLLARY 2.6. *Let f be a group homomorphism from G onto H. Then the following hold:*

(1) If A is a fuzzy p^*-subgroup [resp. p-subgroup] of G, then $f(A)$ is a fuzzy p^*-subgroup [resp. p-subgroup] of H, provided A is f-invariant.

(2) If B is a fuzzy p^*-subgroup [resp. p-subgroup]of H, then $f^{-1}(B)$ is a fuzzy p^*-subgroup [resp. p-subgroup] of G.

PROPOSITION 2.7. *Let f be a group homomorphism from G onto H. And let A and B be fuzzy subgroups of G and H, respectively. Then the following hold:*

(1) $(f(A))_{p^*} \supseteq f(A_{p^*})$ for every prime p.

(2) If either A is f-invariant or G is a divisible group, then $(f(A))_{p^*} = f(A_{p^*})$ for every prime p.

(3) $(f^{-1}(B))_{p^*} = f^{-1}(B_{p^*})$ for every prime p.

THEOREM 2.8. *Let A be a fuzzy subgroup of a group G such that $FO_A(x)$ is finite for all $x \in G$. Then $A_{p^*} = A_p$ for every prime p.*

Thus the notion of A_{p^*} is a generalization of the notion of A_p. Now we show that every fuzzy subgroup A can be written as the intersection of all A_{p^*}.

THEOREM 2.9. *Let A be a fuzzy subgroup of a group G. Then $A = \cap\{A_{p^*} | p$ is a prime$\}$ (briefly, $A = \cap A_{p^*}$).*

3. Fuzzy subgroups and minimal fuzzy p^*-subgroups. In this section, we characterize a fuzzy subgroup as the intersection of its all minimal fuzzy p^*-subgroups.

PROPOSITION 3.1. *Let A be a fuzzy subgroup of a group G. Then A_{p^*} is a fuzzy subgroup of G if and only if $A_{p^*} = A_{(p)^*}$.*

THEOREM 3.2. *Let A be a fuzzy subgroup of a group G. If A_{p^*} is a fuzzy subgroup of G for every prime p, then $A = \cap A_{(p)^*}$.*

THEOREM 3.3. *Let A be a commutative fuzzy subgroup of a group G. Then A_{p^*} is a fuzzy subgroup of G for every prime p.*

COROLLARY 3.4. *Let A be a commutative fuzzy subgroup of a group G. Then $A = \cap A_{(p)^*}$.*

PROPOSITION 3.5. *Let f be a group homomorphism from G onto H. And let A and B be fuzzy subgroups of G and H, respectively. Then the following hold:*

(1) If $A_{(p)^*}$ is f-invariant, then $(f(A))_{(p)^*} = f(A_{(p)^*})$.

(2) $(f^{-1}(B))_{(p)^*} = f^{-1}(B_{(p)^*})$ for every prime p.

THEOREM 3.6. *Let f be a group homomorphism from G onto H. And let A and B be fuzzy subgroups of G and H, respectively. Then the following hold:*

(1) If $A = \cap A_{(p)^*}$, then $f(A) = \cap f(A_{(p)^*}) = \cap(f(A))_{(p)^*}$, provided every $A_{(p)^*}$ is f-invariant.

(2) If $B = \cap B_{(p)^*}$, then $f^{-1}(B) = \cap f^{-1}(B_{(p)^*}) = \cap(f^{-1}(B))_{(p)^*}$.

References

[1] J.-G.Kim, Fuzzy subgroups and minimal fuzzy p-subgroups, J. Fuzzy Math. 2 (1994) 913-921.

[2] J.-G.Kim, Fuzzy subgroups of finite groups and Abelian groups, Math. Japon. 39 (1994) 357-360.

[3] J.-G.Kim, Orders of fuzzy subgroups and fuzzy p-subgroups, Fuzzy Sets and Systems 61 (1994) 225-230.

[4] J.-G.Kim and H.-D.Kim, A characterization of fuzzy subgroups of some Abelian groups, Inform. Sci. 80 (1994) 243-252.