Φ-유계 분산의 비단조 퍼지 측도에 관한 연구

On non-monotonic fuzzy measures of Φ-bounded variation

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Abstract

This paper discuss some properties of non-monotonic fuzzy measures of Φ-bounded variation. We show that there is an example of Φ such that $\mathcal{B}V(X, \mathcal{F})$ is a proper subspace of $\Phi\mathcal{B}V(X, \mathcal{F})$. And also, we prove that $\Phi\mathcal{B}V(X, \mathcal{F})$ is a real Banach space. Furthermore, we investigate some properties of non-monotonic fuzzy Φ-measures.

1. Introduction.

In T. Murofushi, M. Sugeno and M. Machida[1], they discussed non-monotonic fuzzy measures, which are set functions without monotonicity. And also they studied the space of non-monotonic fuzzy measures of bounded variation and investigate some properties of a non-monotonic fuzzy measure of bounded variation. But the fuzzy measure in the sense of Sugeno[3], Q. Z. Wang[4], and Zhong[5] is a monotonic set function. In this paper, we introduce the concept of non-monotonic fuzzy measures of Φ-bounded variation, where $\Phi = \{\phi_n\}$ is a sequence of increasing convex functions, defined on the nonnegative real numbers, such that $\phi_n(0) = 0$ and $\phi_n(x) > 0$ for $x > 0$ and $n = 1, 2, \cdots$.

We say that Φ is a Φ^*-sequence if and only if $\phi_{n+1}(x) \leq \phi_n(x)$ for all n and x, and a Φ-sequence if in additon $\sum_n \phi_n(x)$ diverges for $x > 0$. These definitions were introduced in M. Schramm[2]. In section 2, we introduce non-monotonic fuzzy meaures of Φ-bounded variation and discuss some properties of these fuzzy measures. And also, we prove that $\Phi BV(X,\mathcal{F})$, $\Phi BV_0(X,\mathcal{F})$ are real Banach spaces and that $\Phi BV(X,\mathcal{F})$ is isometrically isomorphic to $\Phi BV_0(X,\mathcal{F})$. Furthermore, we will define non-monotonic fuzzy Φ-measures and discuss some properties of non-monotonic fuzzy Φ-measures.

2. Non-monotonic fuzzy measures of Φ-bounded variation.

Throughout the paper we assume that $(X,\mathcal{F})$ is a measurable space.

Definition 2.1. *A fuzzy measure on $(X,\mathcal{F})$ is a real-valued set function $\lambda:\mathcal{F}\to R^+$ satisfying*

(i) $\lambda(\emptyset)=0$

(ii) $\lambda(A)\le\lambda(B)$ *whenever $A\subset B$ and $A,B\in\mathcal{F}$ where $R^+=[0,\infty)$, the set of nonnegative real numbers.*

Definition 2.2. *A non-monotonic fuzzy measure on $(X,\mathcal{F})$ is a real-valued set function $\mu:\mathcal{F}\to R^+$ satisfying $\mu(\emptyset)=0$.*

In [2], the total variation $V(\mu)$ of μ on X is defined by

$$V(\mu)=\sup\left\{\sum_{i=1}^{n}|\mu(A_i)-\mu(A_{i-1})|\,\middle|\,\emptyset=A_0\subset A_1\subset\cdots\subset A_n=X,\quad\{A_i\}_{i=0}^{n}\subset\mathcal{F}\right\}.$$

A real-valued set function μ is said to be of bounded variation if and only if $V(\mu)<\infty$. And also, we introduce the following definitions of non-monotonic fuzzy measures of Φ-bounded variation and total Φ-variation.

Definition 2.3. *Let $\Phi=\{\phi_n\}$ be either a Φ^*-sequence or a Φ-sequence. For a given real-valued set function $\mu:\mathcal{F}\to R$, the total Φ-variation $\Phi V(\mu)$ of μ on X is defined by*

$$\Phi V(\mu)=\sup\left\{\sum_{i=1}^{n}\phi_i(|\mu(A_i)-\mu(A_{i-1})|)\,\middle|\,\emptyset=A_0\subset A_1\subset\cdots\subset A_n=X,\{A_i\}_{i=0}^{n}\subset\mathcal{F}\right\}$$

A real-valued set function μ is said to be of Φ-bounded variation if and only if $\Phi V(\mu)<\infty$.

Then it is easy to show that if $\Phi=\{\phi_n\}$ and $\phi_n(x)=x$ for each n, then μ is of Φ-bounded variation if and only if μ is of bounded variation.

Definition 2.4. *The family $\{\phi_n\}$ is called the uniformly equicontinuous on R if there is a positive constant M, independent of $n\in N$ and $x,y\in R$, such that*

$$|\phi_n(x)-\phi_n(y)|\le M|x-y| \tag{2-1}$$

We note that if Φ is equicontinuous on R and if $x=|\mu(A_i)-\mu(A_{i-1})|$ and $y=0$, then $\phi_n(|\mu(A_i)-\mu(A_{i-1})|)\le M|\mu(A_i)-\mu(A_{i-1})|$ for $n=1,2,3,\cdots$. Hence, we note that this definition of Φ-bounded variation is a notion of generalized of bounded variation.

Proposition 2.5. *Let $\Phi = \{\phi_n\}$ be as in the Definition 2.3. If in addtion, $\Phi = \{\phi_n\}$ is uniformly equicontinuous, then a monotonic fuzzy measure λ is of Φ-bounded variation.*

Proof: Since $\{\phi_n\}$ is uniformly equicontinuous, there exist a positive constant M such that $\{\phi_n\}$ satisfies (2-1). Hence

$$\begin{aligned}\Phi V(\lambda) &= \sup\left\{\sum_{i=1}^{n}\phi_i\left(|\lambda(A_i)-\lambda(A_{i-1})|\right)\Bigg|\emptyset = A_0\subset A_1\subset\cdots\subset A_n = X, \{A_i\}_{i=0}^{n}\subset\mathcal{F}\right\}\\ &\le \sup\left\{\sum_{i=1}^{n}M|\lambda(A_i)-\lambda(A_{i-1})|\Bigg|\emptyset = A_0\subset A_1\subset\cdots\subset A_n = X, \{A_i\}_{i=0}^{n}\subset\mathcal{F}\right\}\\ &= MV(\lambda) = M\lambda(X) < \infty\end{aligned}$$

Therefore λ is of Φ-bounded variation.

We denote the set of monotonic fuzzy measures on $(X,\mathcal{F})$ by $FM(X,\mathcal{F})$ and the set of non- monotonic fuzzy measures of Φ-bounded variation on $(X,\mathcal{F})$ by $\Phi\mathcal{B}V(X,\mathcal{F})$. Let Φ be as in the definition 2.3 and μ a non-monotonic fuzzy measure on $(X,\mathcal{F})$. Then, definition 2.3 implies that μ is of Φ-bounded variation if and only if there is an $M < \infty$ such that for every finite collection $\{A_i\}_{i=0}^{n}\subset\mathcal{F}$ with $\emptyset = A_0\subset A_1\subset\cdots\subset A_n = X$,

$$\sum_{i=1}^{n}\phi_i\left(|\mu(A_i)-\mu(A_{i-1})|\right) < M$$

Let $\phi_n(x) = \frac{x}{n^2}$ for $n = 1, 2, \cdots$. Then, it is clearly to show that $\mathcal{B}V(X,\mathcal{F})\subset\Phi\mathcal{B}V(X,\mathcal{F})$ and that the converse of implication is not true, that is, the following example 2.6 implies that Φ-bounded variation is some generalization of bounded variation.

Example 2.6. *Let λ be the Lebesque measure on $([0,1],\mathcal{B})$, where $\mathcal{B}$ is the class of all Borel subsets of the unit interval $[0,1]$, and let $h(x) = x\sin\left(\frac{1}{x}\right)$, for all $x\in(0,1]$, $h(0) = 0$, and $\mu = h\cdot\lambda$. Since ϕ is not of bounded variation, it is easy to see that the set function μ is a non-monotonic fuzzy measure that is not of bounded variation. Since $\phi_n(x) = \frac{x}{n^2}$, $\Phi = \{\phi_n\}$ is a Φ^*-sequence and hence, μ is of Φ-bounded variation.*

We recall that if ϕ is an increasing convex function, $\phi(0) = 0$, $x\ge 0$ and $0\le\alpha\le 1$, we have

$$\phi(\alpha x)\le\alpha\phi(x)$$

We define a norm as follows : for every $\mu\in\Phi BV(X,\mathcal{F})$,

$$||\mu||_\Phi = \inf\left\{k>0 : \Phi V\left(\frac{\mu}{k}\right)\le 1\right\}.$$

Proposition 2.7. *Let μ be a non-monotonic fuzzy measure in $\Phi BV(X,\mathcal{F})$. Then, we have that*
(i) $\Phi V\left(\frac{\mu}{\|\mu\|_\Phi}\right) \le 1$
(ii) if $\|\mu\|_\Phi \le 1$, *then* $\Phi V(\mu) \le \|\mu\|_\Phi$

Proof: (i) Take $k > \|\mu\|_\Phi$; then, the definition of $\|\mu\|_\Phi$ implies that for any finite collection

$$\{A_i\}_{i=0}^n \subset \mathcal{F} \quad \text{with} \quad \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = X,$$

we have

$$\sum_{i=1}^n \phi_i\left(\left|\frac{\mu}{k}(A_i) - \frac{\mu}{k}(A_{i-1})\right|\right) \le \Phi V\left(\frac{\mu}{k}\right) \le 1.$$

Thus

$$\sum_{i=1}^n \phi_i\left(\left|\frac{\mu}{\|\mu\|_\Phi}(A_i) - \frac{\mu}{\|\mu\|_\Phi}(A_{i-1})\right|\right) = \lim_{k\to\|\mu\|_\Phi^+} \sum_{i=1}^n \phi_i\left(\left|\frac{\mu}{k}(A_i) - \frac{\mu}{k}(A_{i-1})\right|\right) \le 1,$$

which implies (i).

(ii) Since $\|\mu\|_\Phi \le 1$,

$$\begin{aligned}\sum_{i=1}^n \phi_i(|\mu(A_i) - \mu(A_{i-1})|) &\le \|\mu\|_\Phi \sum_{i=1}^n \phi_i(|\frac{\mu}{\|\mu\|_\Phi}(A_i) - \frac{\mu}{\|\mu\|_\Phi}(A_{i-1})|) \\ &\le \|\mu\|_\Phi\end{aligned}$$

Theorem 2.8. *$\Phi BV(X,\mathcal{F})$ is a normed vector space with a norm $\|\cdot\|_\Phi$*

Proof: Let $\mu,\nu \in \Phi BV(X,\mathcal{F})$ and $c \in R$.

(i) Since ϕ_n is increasing convex for each n, for each $k_1 > k > 0$

$$\begin{aligned}\Phi V\left(\frac{\mu}{k_1}\right) &= \sup \sum_{i=1}^n \phi_i\left(\left|\frac{\mu}{k_1}(A_i) - \frac{\mu}{k_1}(A_{i-1})\right|\right) \\ &= \sup \sum_{i=1}^n \phi_i\left(\frac{k}{k_1}\left|\frac{\mu}{k}(A_i) - \frac{\mu}{k}(A_{i-1})\right|\right) \\ &\le \sup \sum_{i=1}^n \frac{k}{k_1}\phi_i\left(\left|\frac{\mu}{k}(A_i) - \frac{\mu}{k}(A_{i-1})\right|\right) \\ &\le \sup \sum_{i=1}^n \phi_i\left(\left|\frac{\mu}{k}(A_i) - \frac{\mu}{k}(A_{i-1})\right|\right) \\ &= \Phi V\left(\frac{\mu}{k}\right)\end{aligned}$$

Thus, by part (i) of the proposition 2.7, $\left\{k > 0 : \Phi V\left(\frac{\mu}{k}\right) \le 1\right\} = [\|\mu\|_\Phi, \infty)$.

Clearly $\|\mu\|_\Phi = 0$ if $\mu = 0$. Conversely if $\mu \ne 0$, let $A \in \mathcal{F}$ be such that $\mu(A) \ne 0$, then

$$\begin{aligned}\Phi V\left(\frac{\mu}{k}\right) &\ge \phi_1\left(\frac{|\mu(A)|}{k}\right) \\ &\ge \frac{k_1}{k}\phi_1\left(\frac{|\mu(A)|}{k_1}\right) \quad \text{for } k_1 \le k.\end{aligned}$$

As $k \to 0$, $\frac{k_1}{k}\phi_1\left(\frac{|\mu(A)|}{k_1}\right) \to \infty$, and hence $\Phi V(\frac{\mu}{k}) \to \infty$. Thus there is a $k > 0$ so that $\Phi V\left(\frac{\mu}{k}\right) > 1$, and so $\|\mu\|_\Phi \neq 0$;

(ii)

$$\begin{aligned} \|c\mu\|_\Phi &= \inf\left\{k > 0 \middle| \Phi V\left(\frac{c\mu}{k}\right) \leq 1\right\} \\ &= \inf\left\{k > 0 \middle| \Phi V\left(\frac{|c|\mu}{k}\right) \leq 1\right\} \\ &= |c|\|\mu\|_\Phi \end{aligned}$$

(iii) Since each ϕ_i is increasing convex for each $i = 1, 2, 3, \cdots$.

$$\begin{aligned} &\sum_{i=1}^n \phi_i\left(\frac{|(\mu+v)(A_i) - (\mu+v)(A_{i-1})|}{\|\mu\|_\Phi + \|\nu\|_\Phi}\right) \\ &\leq \sum_{i=1}^n \phi_i\left(\frac{|\mu(A_{i-1}) - \mu(A_i)| + |v(A_i) - \nu(A_{i-1})|}{\|\mu\|_\Phi + \|\nu\|_\Phi}\right) \\ &= \sum_{i=1}^n \phi_i\left(\left[\frac{\|\mu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi} + \frac{\|\nu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\right]\frac{|\mu(A_{i-1}) - \mu(A_i)| + |v(A_i) - \nu(A_{i-1})|}{\|\mu\|_\Phi + \|\nu\|_\Phi}\right) \\ &\leq \sum_{i=1}^n \phi_i\left(\frac{\|\mu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\frac{|\mu(A_{i-1}) - \mu(A_i)|}{\|\mu\|_\Phi + \|\nu\|_\Phi} + \frac{\|\nu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\frac{|v(A_{i-1}) - \nu(A_i)|}{\|\mu\|_\Phi + \|\nu\|_\Phi}\right) \\ &\leq \sum_{i=1}^n \left\{\frac{\|\mu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\phi_i\left(\frac{|\mu(A_{i-1}) - \mu(A_i)|}{\|\mu\|_\Phi}\right) + \frac{\|\nu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\phi_i\left(\frac{|v(A_{i-1}) - \nu(A_i)|}{\|\nu\|_\Phi}\right)\right\} \\ &= \frac{\|\mu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\sum_{i=1}^n \phi_i\left(\frac{|\mu(A_{i-1}) - \mu(A_i)|}{\|\mu\|_\Phi}\right) + \frac{\|\nu\|_\Phi}{\|\mu\|_\Phi + \|\nu\|_\Phi}\sum_{i=1}^n \phi_i\left(\frac{|v(A_{i-1}) - \nu(A_i)|}{\|\nu\|_\Phi}\right) \\ &\leq 1 \end{aligned}$$

Hence, $\Phi V\left(\frac{\mu+\nu}{\|\mu\|_\Phi + \|\nu\|_\Phi}\right) \leq 1$. By the Definition of $\|\cdot\|_\Phi$, we have

$$\|\mu + v\|_\Phi \leq \|\mu\|_\Phi + \|\nu\|_\Phi.$$

Therefore, $\|\cdot\|_\Phi$ is a norm on $\Phi BV(X, \mathcal{F})$. And also, (i) and (ii) implies that $\Phi BV(X, \mathcal{F})$ is a vector space.

Theorem 2.9. *$\Phi BV(X, \mathcal{F})$ is a real Banach space with the norm $\|\cdot\|_\Phi$.*

Proof: Let μ and v be fuzzy measure in $\Phi BV(X, \mathcal{F})$ such that $\|\mu - \nu\|_\Phi < \varepsilon < 1$. Since $\frac{\|(\mu-\nu)\|_\Phi}{\varepsilon} < 1$, so, by proposition 2.7,

$$\Phi V\left(\frac{\mu - \nu}{\varepsilon}\right) \leq 1$$

Now for each $A \in \mathcal{F}$,

$$\phi_1\left(\frac{|\mu(A) - \nu(A))|}{\varepsilon}\right) \leq \Phi V\left(\frac{\mu - \nu}{\varepsilon}\right) \leq 1.$$

This implies that if $\{\mu_n\}$ is a Cauchy sequence in this norm $\|\cdot\|_\Phi$, $\{\mu_n(A)\}$ is a Cauchy sequence in R for each $A \in \mathcal{F}$. Thus, there is a real-valued set function μ such that $\mu_m(A) \to \mu(A)$ in R for each $A \in \mathcal{F}$. We note that $\mu_m(\emptyset) = 0$ for all $n = 1, 2, \cdots$. Since $\mu_m(\emptyset) \to \mu(\emptyset)$, $\mu(\emptyset) = 0$. Hence μ is a non-monotonic fuzzy measure on $(X, \mathcal{F})$. Let $\varepsilon > 0$ be given, $\{A_i\}_{i=0}^n$ a finite collection, and suppose that there is some positive integer N_0 such that for any $m, l \geq N_0$, $\|\mu_m - \mu_l\|_\Phi < \varepsilon$, then

$$\begin{aligned}
&\sum_{i=1}^{n} \phi_i \left(\left| \frac{(\mu_m - \mu)(A_i) - (\mu_m - \mu)(A_{i-1})}{\varepsilon} \right| \right) \\
&= \lim_{l \to \infty} \sum_{i-1}^{n} \phi_i \left(\left| \frac{(\mu_m - \mu_l)(A_i) - (\mu_m - \mu_l)(A_{i-1})}{\varepsilon} \right| \right) \\
&\leq \lim_{l \to \infty} \Phi V \left(\frac{\mu_m - \mu_l}{\varepsilon} \right) \leq 1
\end{aligned}$$

Thus $\Phi V(\frac{\mu_m - \mu}{\varepsilon}) \leq 1$, and so $\Phi V(\mu_m - \mu) \leq \varepsilon$. Hence, $\mu_m \to \mu$ in this norm $\|\cdot\|_\Phi$. And also, we have

$$\begin{aligned}
\Phi V(\mu) = \Phi V(\mu - \mu_m + \mu_m) &\leq \Phi V(\mu - \mu_m) + \Phi V(\mu_m) \\
&\leq \varepsilon + \Phi V(\mu_m) < \infty.
\end{aligned}$$

Thus $\mu \in \Phi BV(X, \mathcal{F})$.

Definition 2.10. *For every $\mu \in \Phi BV(X, \mathcal{F})$, we define*

$$\begin{aligned}
|\mu|_\Phi(A) &= \sup \left\{ \sum_{i=1}^{n} \phi_i \left(|\mu(A_i) - \mu(A_{i-1})| \right) \middle| \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = A, \{A_i\}_{i=0}^n \subset \mathcal{F} \right\} \\
\mu_\Phi^+(A) &= \sup \left\{ \sum_{i=1}^{n} \phi_i \left([\mu(A_i) - \mu(A_{i-1})]^+ \right) \middle| \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = A, \{A_i\}_{i=0}^n \subset \mathcal{F} \right\} \\
\mu_\Phi^-(A) &= \sup \left\{ \sum_{i=1}^{n} \phi_i \left([\mu(A_i) - \mu(A_{i-1})]^- \right) \middle| \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = A, \{A_i\}_{i=0}^n \subset \mathcal{F} \right\},
\end{aligned}$$

where $[r]^+ = \max\{r, 0\}$ and $[r]^- = \max\{-r, 0\}$. We call $|\mu|_\Phi$, μ_Φ^+, and μ_Φ^-, the total Φ-variation, positive total Φ-variation, negative total Φ-variation of μ, respectively.

Definition 2.11. *Let μ be a non-monotonic fuzzy measure on $(X, \mathcal{F})$ of Φ-bounded variation and let Φ be either a Φ^*-sequence or a Φ-sequence. Then μ_Φ is defined by*

$$\mu_\Phi(A) = \mu_\Phi^+(A) - \mu_\Phi^-(A), \quad \text{for each } A \in \mathcal{F}$$

In this case, we say that μ_Φ is a non-monotonic fuzzy Φ- measure on $(X, \mathcal{F})$.

We denote the set of monotonic fuzzy measures of Φ-bounded variation on $(X, \mathcal{F})$ by $\Phi FM(X, \mathcal{F})$. From Definitions 2.10 and 2.11, clearly, we obtain the following proposition.

proposition 2.12. *Let $\mu \in \Phi\mathcal{B}V(X,\mathcal{F})$. Then*

(i) μ_Φ^+, $\mu_\Phi^- \in \Phi FM(X,\mathcal{F})$.

(ii) $\mu_\Phi \leq |\mu|_\Phi$.

(iii) $\Phi V(\mu) = \mu_\Phi^+(X) + \mu_\Phi^-(X)$.

We denote the set of a non-monotonic fuzzy Φ-measure on $(X,\mathcal{F})$ by $\Phi\mathcal{B}V_0(X,\mathcal{F})$ and define the norm $\|\cdot\|_0$ by

$$\|\mu_\Phi\|_0 = \mu_\Phi^+(X) + \mu_\Phi^-(X) \quad \text{for each } \mu_\Phi \in \Phi\mathcal{B}V_0(X,\mathcal{F})$$

Proposition 2.13.

(i) $\|\cdot\|_0$ *is a norm on* $\Phi\mathcal{B}V_0(X,\mathcal{F})$

(ii) $(\Phi\mathcal{B}V(X,\mathcal{F}), \|\cdot\|_\Phi)$ *is isometrically isomorphic to* $(\Phi\mathcal{B}V_0(X,\mathcal{F}), \|\cdot\|_0)$.

Proof: Since $\|\mu\|_0 = \|\mu\|_\Phi$ for each $\mu_\Phi \in \Phi\mathcal{B}V_0(X,\mathcal{F})$, the Theorem 2.8 implies (i) and (ii).

We denote the set of monotonic fuzzy Φ-measures on $(X,\mathcal{F})$ by $\Phi FM_0(X,\mathcal{F})$.

Proposition 2.14. $\Phi\mathcal{B}V_0(X,\mathcal{F}) = \Phi FM_0(X,\mathcal{F}) - \Phi FM_0(X,\mathcal{F})$

Proof: Let $\mu_\Phi \in \Phi\mathcal{B}V_0(X,\mathcal{F})$. By the definition 2.11, $\mu_\Phi = \mu_\Phi^+ - \mu_\Phi^-$. The proposition 2.12 (i) implies that $\mu_\Phi \in \Phi FM_0(X,\mathcal{F}) - \Phi FV_0(X,\mathcal{F})$. Since $\Phi\mathcal{B}V(X,\mathcal{F})$ is a real Banach space, by the proposition 2.13 (ii), $\Phi\mathcal{B}V_0(X,\mathcal{F})$ is a real Banach space with the norm $\|\cdot\|_0$. Since $\Phi FM_0(X,\mathcal{F})$ is a subspace of a real Banach space $\Phi\mathcal{B}V_0(X,\mathcal{F})$,$\Phi FM_0(X,\mathcal{F}) - \Phi FM_0(X,\mathcal{F}) \subset \Phi\mathcal{B}V_0(X,\mathcal{F})$.

The variations $|\mu|_\Phi$, μ_ϕ^+ and μ_Φ^- have the following properties.

Proposition 2.15. *Let* μ_Φ, $\nu_\Phi \in \Phi\mathcal{B}V_0(X,\mathcal{F})$ *and* $\alpha \in R$ *with* $|\alpha| < 1$

(i) $|\mu|_\Phi = \mu_\Phi^+ + \mu_\Phi^-$

(ii) $\mu_\Phi^+ = \frac{1}{2}(|\mu|_\Phi + \mu_\Phi)$, $\mu_\Phi^- = \frac{1}{2}(|\mu|_\Phi - \mu_\Phi)$

(iii) $\mu_\Phi^- = (-\mu_\Phi)^-$

(iv) $|\mu|_\Phi = 0 \Longleftrightarrow \mu_\Phi = 0$

(v) $|\alpha\mu|_\Phi \leq |\alpha||\mu|_\Phi$

Proof: Part (i) is essentially the same as proposition 2.12 (iii). Part (ii) is clear from (i) and the definition of μ_Φ. Part (iii) - (iv) are immediate consequences of the definition of μ_Φ.Since ϕ is an increasing convex function, and $|\alpha| < 1$, we have that $\phi(|\alpha|x) \leq |\alpha|\phi(x)$ for all $x \geq 0$. This fact and the definition of μ_Φ imply Part (v).

References

[1]. T. Murofushi, M. Sugeno and Motoya Machida, *Non-monotonic fuzzy measures and the Choquet integral*, Fuzzy Sets and System **64** (1994), 73-86.

[2]. M. Schramn, *Function of Φ-bounded variation and Riemann-Stieltjes integration*, Trans A.M.S **287** (1985), 49-63.

[3]. T. Terano, K. Asai and M. Sugeno, *Fuzzy System Theory and its Applications*, Fuzzy Sets and System (1987).

[4]. Z. Wang and G. J. Klir, *Fuzzy Measures Theory*, New York (1992).

[5]. O. Zhong, *On fuzzy measure and Fuzzy integral on Fussy set*, Fuzzy Sets and System **37** (1990), 77-92.