퍼지수치 측도에 대한 Radon-Nikodym 정리

Radon-Nikodym theorem for fuzzy-number-valued measures

Byung Moon Kil[1] and Yun Kyong Kim[2]

ABSTRACT

Using the representation theorem of fuzzy number, we give the Radon-Nikodym theorem for fuzzy-number-valued measures

1. Introductoin

Set valued and fuzzy set valued measures have received much attention in the last few years because of their usefulness in several applied fields such as mathematical economics and optimal control. The Radon-Nikodym theorem for set valued measures was established by Debreu and Schmeilder[7] and Hiai[10]. For the case of fuzzy set valued measures, it can be found in Ban[2], Puri and Ralescu[14], Stojakovic[17], and so on. In earlier works, in order to introduce a notion of countable additivity, the following concept of sums for a series of fuzzy numbers was used:

$$(\sum_{i=1}^{\infty} u_i)\ (x) = sup \left\{ \wedge_{i=1}^{\infty} u_i(x_i) : x = \sum_{i=1}^{\infty} x_i,\ \ \sum |x_i| < \infty \right\}$$

In this work, we shall introduce a notion of fuzzy number valued measures slightly different from those in the above works and prove a Radon-Nikodym theorem which generalizes the result for real valued measures. This will be accomplished by using the representation theorem of fuzzy number discussed in Goetshel and Voxman[8]

1) Department of Mathematics, Sun Moon University, Asan, Chunganm, Republic of Korea.
2) Department of Mathematics, Dongshin University, Naju, Chonnam, Republic of Korea.

2. Preliminaries

A fuzzy number is fuzzy set $u: R \to [0,1]$ with the properties;

(1) u is normal, i.e., there exists $x \in R$ such that $u(x) = 1$;

(2) u is upper semicontinuous;

(3) u is a convex fuzzy set, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in R$ and $\lambda \in [0,1]$;

(4) $supp\ u = cl\{x \in R\ ;\ u(x) > 0\}$ is compact.

The family of all fuzzy numbers is denoted by E. For a fuzzy set u, we define

$$[u]_\alpha = \begin{cases} \{x : u(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ supp\ u, & \alpha = 0. \end{cases}$$

Then, it is easily established that u is a fuzzy number if and only if

(1) $[u]_\alpha$ is a closed and bounded interval for each $\alpha \in [0,1]$

(2) $[u]_1 \neq \emptyset$.

From this characterization of fuzzy numbers, it follows that a fuzzy number u is completely determined by the end points of the intervals $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$.

Theorem 2.1 ([8]). For $u \in E$, *denote* $u^-(\alpha) = u_\alpha^-, u^+(\alpha) = u_\alpha^+$. Then

(1) $u^-(\alpha)$ *is a bounded increasing function on* $[0,1]$;

(2) $u^+(\alpha)$ *is a bounded decreasing funstion on* $[0,1]$;

(3) $u^-(1) \leq u^+(1)$;

(4) $u^-(\alpha)$ and $u^+(\alpha)$ *are left continuous on* $(0,1]$ *and right continuous at* 0;

(5) *If* $u^-(\alpha)$ and $u^+(\alpha)$ *satisfy above* (1)~(4), *then there exists a unique* $v \in E$ *such that* $v_\alpha^- = u^-(\alpha)$, $v_\alpha^+ = u^+(\alpha)$

The above theorem implies that we can identify a fuzzy number u with the parametrized representation $\{(u_\alpha^-, u_\alpha^+) \mid 0 \leq \alpha \leq 1\}$. We define a meric d on E by

$$d(u,v) = sup_{0 \leq \alpha \leq 1} d_H([u]_\alpha, [v]_\alpha),$$

where d_H is the Hausdorff meric defined as

$$d_H([u]_\alpha, [v]_\alpha) = \max(|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|).$$

Also, $d(u, I_{\{0\}})$ will be denoted by $\| u \|$.

3. Integrals of fuzzy number valued functions

Throughout this paper, (Ω, Σ, μ) denotes a complete σ-finite measure space. If $F : \Omega \to E$ is a fuzzy number valued function and B is a subset of R, then $F^{-1}(B)$ denotes the fuzzy subset of Ω defined by $F^{-1}(B)(\omega) = sup_{x \in B} F(\omega)(x)$ for every $\omega \in \Omega$. The fuzzy number valued function $F : \Omega \to E$ is called measurable if for every closed subset B of R the fuzzy set $F^{-1}(B)$ is measurable when considered as a function from Ω to [0,1]. This concept of measurability for fuzzy set valued functions was introduced by Butnariu[3] as a natural generalization of measurable multifunctions. Kaleva[11] defined $F : \Omega \to E$ to be strongly measurable if for each $\alpha \in [0,1]$ the set valued function $F_\alpha : \Omega \to I(R)$ defined by $F_\alpha(\omega) = [F(\omega)]_\alpha$ is measurable, where I(R) is the set of all closed bounded intervals on R endowed with the topology generated by the Hausdorff metric d_H. The next theorem shows that the above definitions are equivalent.

Theorem 3.1 For $F : \Omega \to E$, $F(\omega) = \{(F_\alpha^-(\omega), F_\alpha^+(\omega)) \mid 0 \leq \alpha \leq 1\}$, *the following conditions are equivalent.*

(1) *F is measurable.*

(2) *F is strongly measurable.*

(3) *For each $\alpha \in [0,1]$, F_α^- and F_α^+ are measurable.*

Definition 3.2. *A measurable function* $F: \Omega \to E$, $F(\omega) = \{F_\alpha^-(\omega), F_\alpha^+(\omega) \mid 0 \le \alpha \le 1\}$ *is called integrable if for each* $\alpha \in [0,1]$, F_α^- and F_α^+ *are integrable, or equivalently, if* F_0^- and F_0^+ *are integrable. In this case, the integral of F over* $A \in \Sigma$ *is defined by*

$$\int_A F\, d\mu = \left\{ \left(\int_A F_\alpha^-\, d\mu, \int_A F_\alpha^+\, d\mu \right) \mid 0 \le \alpha \le 1 \right\}.$$

It follows immediately that F is integrable if and only if the real valued function $\omega \mapsto \| F(\omega) \|$ is integrable. Note that in [2,5,11,13,16] the terminology *integrably bounded* was used instead.

The following theorem first obtained by Puri and Ralescu[13] for the case of nonatomic measure spaces extends the classical *Lebesgue dominated convergence theorem.*

Theorem 3.3. *Let* $\{F_n, n \ge 1\}$ *and* F *be measurable fuzzy number valued functions such that* $F_n(\omega) \xrightarrow{d} F(\omega)$ *for almost every* $\omega \in \Omega$. *If there exists an integrable function* $h: \Omega \to R$ *such that for all n*

$$\| F_n(\omega) \| \le h(\omega),$$

then F is integrable and

$$\int F_n\, d\mu \xrightarrow{d} \int F\, d\mu.$$

4. Radon-Nikodym theorem for fuzzy number valued measures

Definition 4.1. *Let* $\{u_n\}$ *be a sequence of fuzzy numbers in* E *and* $u \in E$. *The series* $\sum_{n=0}^{\infty} u_n$ *is said to be convergent to* u *if* $d(\sum_{i=1}^{n} u_i, u) \to 0$ *as* $n \to \infty$.

In this case, we call u *the sum of the series* $\sum_{n=1}^{\infty} u_n$ *and write* $u = \sum_{n=1}^{\infty} u_n$.

It follows immediately that the series $\sum_{n=1}^{\infty} u_n$ converges to u if and only if

$$\sum_{i=1}^{n}(u_i)_\alpha^- \to u_\alpha^- \qquad \text{and} \qquad \sum_{i=1}^{n}(u_i)_\alpha^+ \to u_\alpha^+$$

uniformly in $\alpha \in [0,1]$. Furthermore, if $u_n = \{(u_{n,\alpha}^-, u_{n,\alpha}^+) \mid 0 \le \alpha \le 1\}$ are fuzzy numbers such that $\sum_{n=1}^{\infty} u_{n,\alpha}^- = u_\alpha^-$ and $\sum_{n=1}^{\infty} u_{n,\alpha}^+ = u_\alpha^+$ converge uniformly in α, then $u = \{(u_\alpha^-, u_\alpha^+) \mid 0 \le \alpha \le 1\}$ is a fuzzy number and $u = \sum_{n=1}^{\infty} u_n$.

Definition 4.2 *A fuzzy number valued measure is a set function* $M : \Sigma \to E$ *with properties*

(1) $M(\emptyset) = I_{\{0\}}$

(2) $M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$ *for every sequence* $\{A_n\}$ *of pairwise disjoint elements of* Σ.

Note that a set function $M : \Sigma \to E$, $M(A) = \{(M_\alpha^-(A), M_\alpha^+(A)) \mid 0 \le \alpha \le 1\}$ is a fuzzy number valued measure if and only if

(1) For each $\alpha \in [0,1]$, M_α^- and M_α^+ are real valued measures,

(2) The families $\{M_\alpha^- \mid 0 \le \alpha \le 1\}$ and $\{M_\alpha^+ \mid 0 \le \alpha \le 1\}$ of measures are both uniformly countably additive, that is for any sequence $\{A_n\}$ of pairwise disjoint elements of Σ, the convergence of $M_\alpha^-(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M_\alpha^-(A_n)$ is uniform in $\alpha \in [0,1]$, and likewise for $\{M_\alpha^+ \mid 0 \le \alpha \le 1\}$.

Definition 4.3. *If* m *is a classical measure in* Σ, *and* M *is a fuzzy number valued measure, then* M *is called absolutely continuous with respect to* m *if* $m(A) = 0$ *implies* $M(A) = I_{\{0\}}$.

Theorem 4.4. *If* $F : \Omega \to E$ *is integrable, then the set function defined by*

$$M(A) = \int_A F\, d\mu, \qquad A \in \Sigma,$$

is a fuzzy number valued measure which is absolutely continuous with respect to μ.

The next theorem is the converse of the above theorem which generalizes the classical *R-adon-Nikodym theorem.*

Theorem 4.5. *If a fuzzy number valued measure* $M : \Sigma \to E$ *is absolutely continuous with respect to* μ, *then there exists a unique integrable function* $F : \Omega \to E$ *such that for all* $A \in \Sigma$,

$$M(A) = \int_A F\, d\mu.$$

REFERENCES

1. R. J. Aumann, *Integrals of set-valued functions,* J. Math. Anal. Appl. 12(1965), 1-12.
2. J. Ban, *Radon-Nikodym theorem and conditional expectation of fuzzy-valued measures and variables,* Fuzzy Sets and Systems 34(1990), 383-392.
3. D. Butnariu, *Measurability concepts for fuzzy mappings,* Fuzzy Sets and Systems 31(1989), 77-82.
4. W. Congxin and M. Ming, *Embedding problem of fuzzy number space: part I,* Fuzzy Sets and Systems 44(1991), 33-38.
5. W. Congxin and M. Ming, *Embedding problem of fuzzy number space: part II,* Fuzzy Sets and Systems 45(1992), 189-202.
6. G. Debreu, *Integration of correspondences,* Proc. 5th Berkeley Symp. Math. Statist. Prob.2 (1966), 351-372.
7. G. Debreu and D. Schmeidler, *The Radon-Nikodym derivarive of a correspondence,* Proc. 6th Berkeley Symp. Math. Statist. Prob. 2(1972), 41-56.
8. R. Goetschel and W. Voxman, *Elementary fuzzy calculus,* Fuzzy Sets and Systems 18 (1986), 31-43.
9. F. Hiai and H. Umegaki, *Integrals, conditional expectations and martingales of multivalued functions,* J. Multivariate Anal. 7(1977), 149-182.

10. F. Hiai, *Radon-Nikodym theorems for set-valued measures,* J. Multivariate Anal. 8(1978), 96-118.

11. O. Kaleva, *Fuzzy differential equations,* Fuzzy Sets and Systems 24(1987), 301-317.

12. N. Papageorgiou, *On the theory of Banach space valued multifunctions part I: Integration and conditional expectation,* J. Multivariate Anal. 17(1985), 185-206.

13. M. Puri and D. Ralescu, *Fuzzy random variables,* J. Math. Anal. Appl. 114(1986), 409-422

14. M. Puri and D. Ralescu, *Convergence theorem for fuzzy martingales,* J. Math. Anal. Appl. 160(1991), 107-122.

15. M. Stojakovic, *Fuzzy conditional expectation,* Fuzzy Sets and Systems 52(1992), 53-60.

16. M. Stojakovic, *Fuzzy random variables, expectation and martingales,* J. Math. Anal. Appl. 184(1994), 594-606.

17. M. Stojakovic, *Fuzzy valued measure,* Fuzzy Sets and Systems 65(1994), 95-104.