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비틀림 하중을 받는 얇은 빔의 동적 불안정성에 관한 연구

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Study on the Dynamic Torsional Instability of a Thin Beam

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Introduction

In recent years, many researcher have been interested in the stability of a thin beam.

Among them, Pai and Nayfeh[1] had investigated the nonplanar motion of the cantilever beam under lateral base excitation and chaotic motion, but this study is associated with internal resonance, i.e. one to one resonance.

Also Cusumano[2] had made an experiment on a thin beam, called Elastica, under bending loads. In this experiment, he had shown that there exists out-of-plane motion, involving the bending and the torsional mode.

Pak *et al.*[3] verified the validity of Cusumano's experimental works theoretically and defined the existence of Non-Local Mode(NLM), which is came out due to the instability of torsional mode and the corresponding aspect of motions by using the Normal Modes. Lee[4] studied on a thin beam under bending loads and investigated the routes to chaos by using forcing amplitude as a control parameter.

In this paper, we are interested in the motion of a thin beam under torsional loads. Here the form of force based on the natural forcing function is used. Consequently, it is found that small torsional loads result in instability and in case that the forcing amplitude is increasing gradually, the motion appears in the form of dynamic double potential well, finally leads to complex motion.

This phenomenon is investigated through the poincaré map and time response. We also check that Harmonic Balance Method(H.B.M) is a suitable tool to calculate the bifurcated modes.

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Equations of motion.

Equations of motion are derived from the following simple mechanical analogue to a thin beam.

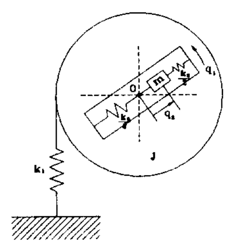

Fig 1. A simple mechanical analogue to a thin beam.

The kinetic energy T and potential energy V are obtained

$$T=\frac{m}{2}(q_2^2\,{q'_1}^2+{q_2'}^2)+\frac{J}{2}{q'_1}^2$$
$$V=\frac{1}{2}(k_1 q_1^2+k_2 q_2^2) \qquad '=\frac{d}{d\tau} \tag{1}$$

By nondimensionalization, the non-dimensional kinetic energy T and potential energy V are as follows

$$T=\frac{1}{2}(1+\mu y^2)\dot{x}^2+\frac{1}{2}\dot{y}^2$$
$$V=\frac{1}{2}(p^2x^2+y^2) \qquad (2)$$

where, $x=\sqrt{J\,k_1/m}\;q_1$ is the nondimensionalized torsional displacement, $y=\sqrt{k_1}\;q_2$ is the nondimensionalized bending displacement, $\mu=m/J\,k_1$ is coupling parameter and p the frequency ratio.

By Lagrange equation, the equations of motion are written as

$$(1+\mu y^2)\ddot{x} + 2\mu y\dot{y}\dot{x} + p^2x = 0$$
$$\ddot{y} - \mu\dot{x}^2 y + y = 0 \qquad (3)$$

Free vibration and Bifurcation modes

It is easily understood from Eq.(3) that there are two trivial solutions $x = 0$, $y = 0$.

One motion is the torsional mode, which is equal to the condition $x \neq 0$ and $y = 0$ as shown in Fig 2. Another motion is the bending mode, $x = 0$ and $y \neq 0$ in Fig. 2. These are straight lines in the x-y configuration space, hence these are called similar normal modes. Besides, there are other types of periodic solutions from perturbation analysis. For example, Nonlinear Normal Mode(NNM), Elliptic Orbit(EO) and Non-Local Mode(NLM) as shown in Fig. 2. These all are involved both the torsional mode and the bending modes. Especially in case of NLM, it is characterized that the period of the bending mode is twice that of the torsional mode.

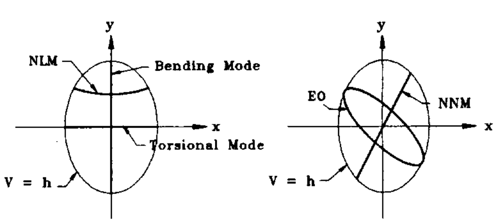

Fig 2. Various periodic solutions of elastica.

To study the stability of torsional mode, perturbation ζ is superimposed, resulting in

$$x = A\sin pt$$
$$y = 0+\zeta \qquad (4)$$

Substituting Eq.(4) into Eq.(3), the resulting equation is turned to be the form of standard Mathieu's equation.

$$\zeta'' + (\delta + 2\varepsilon\cos 2\tau)\,\zeta = 0 \qquad (5)$$

where, $\varepsilon=-\frac{1}{4}\mu A^2$, $\delta=\frac{1}{p^2}+2\varepsilon$, $\tau=pt$, $'=\frac{d}{d\tau}$

The corresponding stability chart is shown in Fig. 3.

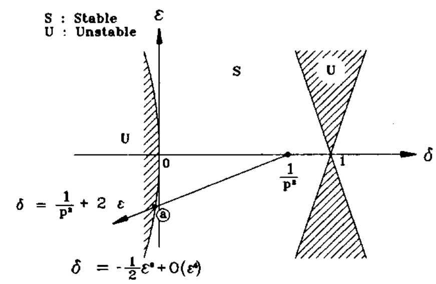

Fig 3. Stability chart for the torsional mode

The stability is changed when $\delta=\frac{1}{p^2}+2\varepsilon$ crosses the transition curve $\delta=-\frac{1}{2}\varepsilon^2+O(\varepsilon^4)$.

The total energy H at the point ⓐ, denoted h_o , can be calculated from the stability chart. After stability change happen, the eigenfunction corresponds to transition curve is seed for bifurcated mode.

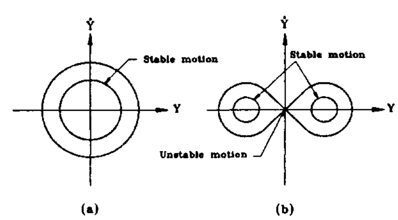

Fig 4. Poincaré map of torsional mode
(a) pre-bifurcation (b) pro-bifurcation

Hence Harmonic Balance Method(HBM) can be utilized and the form of solution are as follows

$$x = A \sin \omega t$$
$$y = B + C \cos 2\omega t \qquad (6)$$

Substituting Eq.(6) into Eq.(3) to obtain

$$A[\ p^2 - w^2 - \mu w^2(B^2 + BC + \frac{1}{2}C^2)] = 0$$
$$B - \frac{1}{2}\mu w^2 A^2 (B + \frac{1}{2}C) = 0 \qquad (7)$$
$$(1 - 4w^2)C - \frac{1}{2}\mu w^2 A^2 (B + C) = 0$$

Relation between frequency ω and amplitudes can be obtained from Eq.(7) as depicted in Fig. 5.

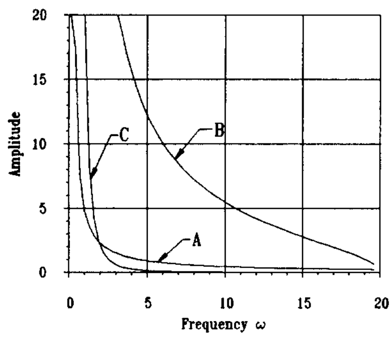

Fig 5. Backborn curve of free vibration

This curve is called backborn curve. It indicates that the amplitudes are inversely proportional to frequency ω.

According as the total energy H increases, the motion goes along the the arrowed line in Fig. 3.

At the point ⓐ, total energy H is equal to h_o . For the value of H < h_o, the system is in the state of the pure torsional mode, which shown as depicted in Fig 6. (a). The other hand for H > h_o , though this area is unstable region, shaded area in Fig. 3, the new stable motion(NLM) can be observed as shown in Fig. 6(b).

Forced Vibration

The equation to the forced vibration of thin beam is

$$(1 + \mu y^2)\ddot{x} + 2\mu y \dot{y} \dot{x} + p^2 x = F_o \sin \Omega t \qquad (8)$$
$$\ddot{y} - \mu \dot{x}^2 y + y = 0$$

where, Ω is the excitation frequency and F_o is

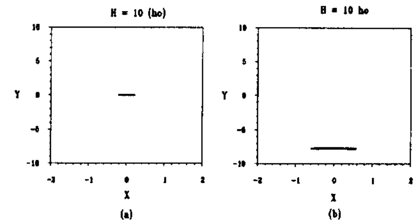

Fig 6. Existence of Nonlocal mode on configuration space

(a) pre-bifurcation (b) post-bifurcation

excitation amplitude.

Substituting Eq.(6), in which ω is replaced by Ω into Eq.(8) and collecting the coefficients of harmonic terms, the following equations can be obtained.

$$A[\ p^2 - \Omega^2 - \mu\Omega^2(B^2 + BC + \frac{1}{2}C^2)] = F_o \qquad (9\text{-a})$$
$$B - \frac{1}{2}\mu\Omega^2 A^2 (B + \frac{1}{2}C) = 0 \qquad (9\text{-b})$$
$$(1 - 4\Omega^2)C - \frac{1}{2}\mu\Omega^2 A^2 (B + C) = 0 \qquad (9\text{-c})$$

From Eq.(9), there are two cases according to the value of B and C. One is that B and C is equal to zero, which means the pure torsional mode. The other case is that B and C is not equal to zero, which means the coupled torsional bending mode.

In case of the pure torsional mode, the pure torsional amplitude $A = \frac{F_o}{p^2 - \Omega^2}$ can easily be obtained from Eq.(9-a).

If the linearized frequency ratio p is equal to the excitation frequency Ω, it means that the system is in the state of resonance.

Also, in case of the coupled torsional bending mode, B and C in Eq.(9-b) and Eq.(9-c) must have a non-trivial solution(i.e. $B \neq 0$, $C \neq 0$).

So Eq.(9-b) and Eq.(9-c) are rewritten as follows,

$$(1 - \frac{\alpha}{2})B - \frac{\alpha}{4}C = 0$$
$$-\frac{\alpha}{2}B + (1 - 4\Omega^2 - \frac{\alpha}{2})C = 0 \qquad (10)$$

where, $\alpha = \mu\Omega^2 A^2$.

To have non-trivial solutions, the determinant Δ

$$\Delta = \begin{vmatrix} 1-\frac{a}{2} & -\frac{a}{4} \\ -\frac{a}{2} & 1-\frac{a}{2}-4\Omega^2 \end{vmatrix} \qquad (11)$$

must vanish.

The value that determinant Δ is equal to zero is the value that the arrowed line passes the point ⓐ in Fig. 3. If the exact transition curve, having higher order term than second order in ε, is used, determinant Δ is much closer to zero. Hence the following relations can be obtained from Eq. (9).

$$A^2 = \frac{8(-4\Omega^2 - \sqrt{(4\Omega^2-1)^2+1})}{\left\{1-\frac{1}{2}(-4\Omega^2-\sqrt{(4\Omega^2-1)^2+1}\ )^2\right\}\mu}$$

$$B = \frac{-4\Omega^2 - \sqrt{(4\Omega^2-1)^2+1}}{2}\, C$$

$$C = \sqrt{\frac{(p^2 - w^2 - F_o/A)}{\mu\Omega^2 S}} \qquad (12)$$

$$\textit{where},\quad S = \frac{1}{4}(-4\Omega^2 - \sqrt{(4\Omega^2-1)^2+1}\ )^2 + \frac{1}{2}(-4\Omega^2 - \sqrt{(4\Omega^2-1)^2+1}\ ) + \frac{1}{2}$$

Sweeping excitation frequency Ω, relation between amplitudes and the excitation frequency, can be obtained as shown in Fig. 7. Such a graph is called frequency response curve.

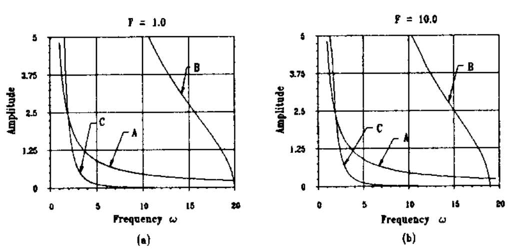

Fig 7. Frequency response curve

From Eq.(12), the bending amplitude B and C vary according to the forcing amplitude F_o. But as the torsional amplitude A is a function of Ω, so A is independent of the forcing amplitude F_o.

Also when the forcing amplitude is very small, frequency response curve is similar to backborn curve of free vibration.

Stability Analysis

To study the stability of the torsional mode, ζ is perturbed in y ,

$$\begin{aligned} x &= \frac{F_o}{p^2 - \Omega^2} \sin \Omega t \\ y &= 0 + \zeta \end{aligned} \qquad (13)$$

Substituting Eq.(13) into Eq.(8), retaining only linear terms in perturbation ζ, resulting in

$$\zeta'' + (\ \delta + 2\varepsilon \cos 2\tau\ )\ \zeta = 0$$

where, $\delta = \frac{1}{\Omega^2} + 2\varepsilon$, $\varepsilon = -\frac{\mu}{4}\left(\frac{F_o}{p^2-\Omega^2}\right)^2$

This is the type of standard Matheiu's equation, which is exact equal to that of free vibration if starting point $\frac{1}{p^2}$ is replaced by $\frac{1}{\Omega^2}$ in Fig. 3

Using F as control parameter, the corresponding amplitude can be obtained as shown in Fig. 8

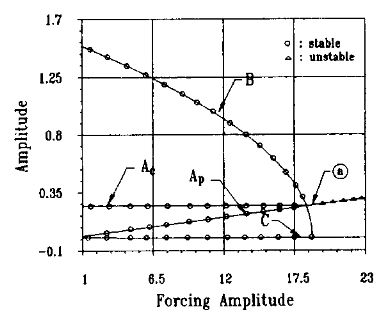

Fig 8. Bifurcation diagram by using F as parameter
Ap : amplitude of pure torsional mode
Ac : amplitude of coupled mode

There is a stability change at point ⓐ as depicted in Fig. 8. In case that pure torsional motion is used as initial condition, for $F > F_ⓐ$, $F_ⓐ$ indicated the forcing amplitude at point ⓐ, the pure torsional motion is in unstable state. The value of the forcing amplitude at point ⓐ in the stability chart is equal to that of Fig. 8. Finally the corresponding forcing amplitude at point ⓐ is to be $F_{ⓐ} = \sqrt{\dfrac{2(p^2-\Omega^2)^2}{\mu\Omega^2}}$.

Numerical simulations, using Runge-Kutta 4th algorithm, are accomplished to show that it is in accord with analytic results.

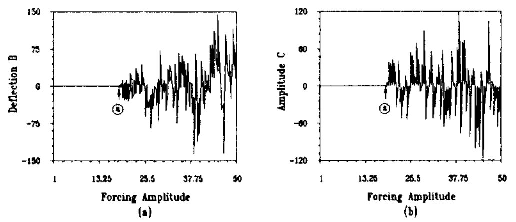

Fig 9. Diagram for transition from the pure torsional mode to the bending mode
(a) Bending Deflection (b) Bending Amplitude

The result of Fig 8. is in accord with that of Fig 9. Poincaré map can be used to determine the stability of the torsional mode. Fig. 10 (a) represents the periodic motion, which has one period motion but in Fig. 10 (b) the exact period can't be compute, which means the unstable motion.

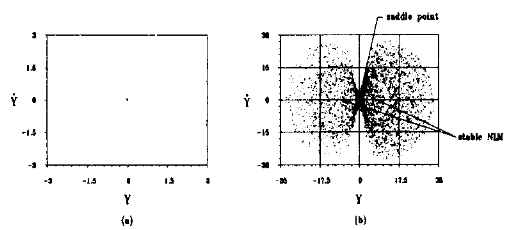

Fig. 10 Poincaré Map for pre- and pro-stability change

In unstable region, the motion is sometimes represented as the type of dynamic double potential well behavior as shown Fig. 11.

Dynamic double potential well behavior is that the thin beam move from left hand to right hand or vice versa.

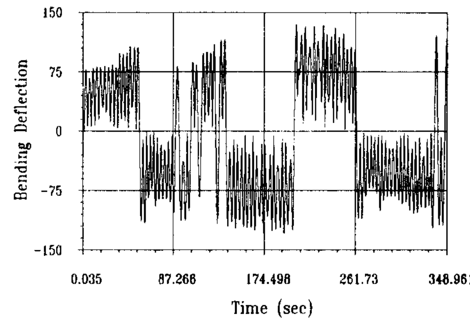

Fig. 11 Behavior of dynamic double potential well

Remarks

It is found that small torsional loads can make the motion of beam to be unstable. Also we know that there exist two types of motions, one is the pure torsional mode and the other is the coupled torsional bending mode. When the pure torsional motion passes the point ⓐ in Fig. 8, stability change happen. And this make the pure torsional motion to be unstable. The other hand in case that coupled torsional bending mode are regarded as initial conditions, the corresponding behaviors exist in region prior to point ⓐ in Fig. 8. The type of dynamic double potential well behaviors can be also observed in Fig. 11.

It is also checked that H.B.M method is a suitable tool in order to compute the bifurcated mode.

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