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위그너-빌 분포함수의 계산시 창문함수의 적용에 의한 바이어스 오차

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(The Bias Error due to Windows for the Wigner-Ville Distribution Estimation)

(Yon-Kyu Park, Yang-Hann Kim)

1. INTRODUCTION

Although spectrum analysis using FFT algorithm has been extensively used to analyze mechanical signatures such as vibration and acoustic signals, there has been increasing demands of other processing methods which not only provide other representations of signal of interest but also appropriately process non-stationary signals. A non-stationary signal in time domain could be regarded as that of a signal that changes its frequency with respect to time, or more generally, its statistical characteristics change in terms of time. The time-frequency analysis of signals is one of the methods for analyzing non-stationary signal.

There are various time-frequency analysis tools; instantaneous frequency, short-time Fourier transform(STFT), wavelet transform, and Wigner-Ville distribution(WVD) are some examples. Instantaneous frequency shows frequency variation along time, therefore it can be used to analyze the transient signal with conventional amplitude-time representation of signal[1, 2]. On the other hand, WVD can not only show the frequency variation with respect to time but also exhibit energy contents of the signal along with time and frequency, and produces some additional components to the signal, due to its estimation.

Recently, WVD has been regarded as a useful tool and applied to various types of mechanical noise and vibration signals; some of the applications are wave decomposition in a beam and characterization of dispersion relation[3, 4]. WVD has been studied in terms of many properties in time and frequency domain; its uniqueness condition, shift and modulation properties, the relation between WVD and instantaneous frequency or group delay, and symmetric properties, etc [5, 6]. It is also common practice to use analytic signal to avoid aliasing problem of WVD in frequency domain[5, 7] and also to smooth the WVD to reduce the variance and to eliminate the possible negative value of WVD[5, 8].

For practical transient signal processing, time data cannot be infinite but must have finite length, therefore the length of time dependent autocorrelation function varies with time, the size of Fourier transform which is to be carried out at each time to get WVD; the window size of Fourier transform, also varies with time. In this paper, the effects of the time varying window length on the WVD are investigated. The bias error and the frequency resolution of WVD are influenced by the time varying window length.

The aforementioned smoothed WVD can be obtained by two-dimensional convolution along time and frequency axes, of true WVD and smoothing window function. The error bound caused by this inherent smoothing algorithm is derived by applying the method proposed by A. Papoulis[9]. The error bound is found explicitly in terms of the smoothing window function; in the case of using a Gaussian window the error is found to be an infinite summation of differential operators. For cases other than Gaussian window, the error bound of smoothed WVD is also analytically obtained.

2. EFFECTS OF FINITE RECORD

Wigner-Ville distribution which was introduced in 1932[10], considers a transformation of a wave function into probability function of the simultaneous values of n independent variables and n momenta. If one considers only an instant of time and frequency then, WVD, W (t, ω) can be written as

$$W(t,\omega) = \int_{-\infty}^{\infty} s(t+\frac{\tau}{2})\, s^*(t-\frac{\tau}{2})\, e^{-j\omega\tau}\, d\tau \qquad (1)$$

which represents the instantaneous energy of a signal along its frequency axis when one sees Eq.(1) at fixed or arbitrary time, or amplitude variation of signal along time axis if it is projected from an arbitrary frequency. In Eq.(1), s(t) is an analytic signal, * denotes complex conjugate, and t, ω are time and frequency respectively. For simplicity, let $s(t+\frac{\tau}{2})\, s^*(t-\frac{\tau}{2})$ be c(t, τ), the time dependent

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autocorrelation function, then Eq.(1) becomes

$$W(t, \omega) = \int_{-\infty}^{\infty} c(t, \tau)\, e^{-j\omega\tau}\, d\tau \quad (2)$$

Because of the finite length of the signal, the estimation of WVD involves the use of a window; in other words, the infinite integration is to be finite integration. Therefore, Eq.(2) has to be rewritten as

$$W_D(t, \omega) = \int_{-\infty}^{\infty} c(t, \tau)\, d(t, \tau)\, e^{-j\omega\tau}\, d\tau \quad (3)$$

where, $W_D(t, \omega)$ is the windowed WVD, and $d(t, \tau)$ is the window function to express the finite length of the signal due to the truncation of the signal in practice. It is therefore possible to infer that the window function $d(t, \tau)$, especially the window length is a function of time. If the length of the signal is L, then $s(t) = 0$ for $t < 0$ and $t > L$. Henceforth, from the definition of $c(t, \tau)$ and from Eq.(3), $c(t, \tau)$ has non-zero value when the conditions $2t - 2L < \tau < 2t$ and $-2t < \tau < 2L - 2t$ are satisfied. Fig.1 depicts the region of τ and t. It is noteworthy that the range of $d(t, \tau)$; the window size ($M(t)$), must be within the non-zero range of $c(t, \tau)$. Therefore, from Fig.1, one can resolve that

$$M(t) = \begin{cases} 4t & , 0 < t < \frac{L}{2} \\ 4L - 4t & , \frac{L}{2} < t < L \end{cases} \quad (4)$$

The maximum of $M(t)$ is 2L at $t = L/2$. As well known, resolution in frequency domain is inversely proportional to window size, in this case time varying window, $M(t)$, therefore the resolution is also a function of an instant time. When $M(t)$ is large, the resolution is high. Especially at $t = L/2$, the resolution is highest. On the contrary, when $M(t)$ is small, the resolution is low. As an extreme case, at $t = 0$ or $t = L$, the resolution becomes lowest and the resolution band is infinity.

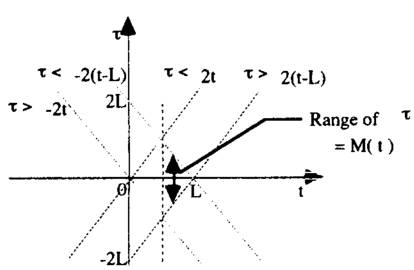

Fig.1 The range of τ in which the time-dependent autocorrelation function has non-zero value.

From these observations, it is now obvious that the relation between the use of window and the bias error due to the window on the estimation of WVD must be conveyed in more details. To see the effects of finite length on the bias error, let's start with the window having time varying length ($M(t)$), in time domain first.

Eq.(3) can be rewritten as a convolution in the frequency domain.

$$W_D(t, \omega) = W(t, \omega) \otimes D(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(t, \omega - \omega')\, D(t, \omega')\, d\omega' \quad (5)$$

where, $\otimes$ denotes the convolution in frequency domain, and $W(t, \omega)$, $D(t, \omega)$ are the Fourier transform of $c(t, \tau)$, $d(t, \tau)$ respectively. As mentioned before, $d(t, \tau)$ is a function of time as well as its length. Therefore, $D(t, \omega)$ is also a function of time, henceforth $W_D(t, \omega)$, which is the convolution of $W(t, \omega)$ and $D(t, \omega)$, is also a function of time, and is related to the time varying window length.

If $W(t, \omega - \omega')$ is assumed to be continuous up to second derivative, then it is possible to expand $W(t, \omega - \omega')$ around $W(t, \omega)$ as a Taylor series, and can be approximated as

$$W(t, \omega - \omega') \cong W(t, \omega) - \omega' W_\omega(t, \omega) + \frac{1}{2} \omega'^2 W_{\omega\omega}(t, \omega) \quad (6)$$

where the subscript denotes independent variables of differentiation. In fact, one could generalize Eq.(6) by including higher order derivatives of WVD with respect to frequency; a similar procedure will be shown in the next section for two-dimensional smoothing, which implies two-dimensional convolution along time and frequency axes, but for simplicity, the assumption of Eq.(6) is to be kept.

Using Eq.(6) and the fact that $D(t, \omega)$ can be assumed to be an even function of ω without the loss of generality, the windowed WVD, $W_D(t, \omega)$ can be obtained as

$$W_D(t, \omega) \cong W(t, \omega) + \frac{1}{4\pi} W_{\omega\omega}(t, \omega) \int_{-\infty}^{\infty} \omega^2 D(t, \omega')\, d\omega' \quad (7)$$

Now the bias error due to the window can be defined as the difference between true and windowed WVD and can be obtained as

$$E(t, \omega) = W_D(t, \omega) - W(t, \omega) \cong \frac{1}{4\pi} W_{\omega\omega}(t, \omega) \int_{-\infty}^{\infty} \omega^2 D(t, \omega')\, d\omega' \quad (8)$$

From Eq.(8) one can conclude that the bias error due to the window is proportional to the second moment of $D(t, \omega)$ and second derivative of $W(t, \omega)$ with respect to frequency. In practice, this means that a spiky WVD in frequency tends to produce large bias error but can be reduced if one uses the window whose second moment with respect to frequency is small; as an extreme case, for a sinusoidal signal, one has to use dirac delta type window in frequency domain, which means infinite flat window in time domain, therefore has infinite data length. To see the more specific relation between the error bound and the window size M(t), the relation between the window size and its second moment in frequency must be conveyed. As mentioned briefly, if the window size is large, then D(t, ω), the Fourier transform of the window is narrow; its second moment is small. Therefore, if M(t) is large, then the bias error will be small, and if M(t) is small, then the bias error will be large.

The effect of window on two-dimensional convolution for having smoothed WVD will now be examined.

3. BIAS ERROR DUE TO SMOOTHING

A smoothed WVD can be calculated by the two-dimensional convolution between true WVD and the smoothing window function, that is

$$W_s(t, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t-t', \omega-\omega')\, G(t', \omega')\, dt'\, d\omega' \quad (9)$$

where, G(t, ω) is the two-dimensional window function. In this convolution procedure, the lengths of the smoothing window are fixed along the time and frequency axes, unlike the case of previous section, where the window in time domain has time varying length. If W(t, ω) is continuous and differentiable in any order of interest, then W(t - t', ω - ω') can be expanded around W(t, ω) as

$$W(t-t', \omega-\omega') = W(t, \omega) - \left\{ t' W_t(t, \omega) + \omega' W_\omega(t, \omega) \right\} + \frac{1}{2!}\left\{ t'^2 W_{tt}(t, \omega) + 2t'\omega' W_{t\omega}(t, \omega) + \omega^2 W_{\omega\omega}(t, \omega) \right\} + \cdots. \quad (10)$$

Using Eq.(10), assuming that G(t, ω) is an even function of t and ω without the loss of generality, and having the energy is to be conserved during the smoothing, i.e. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(t, \omega)\, dt\, d\omega = 1$, then the smoothed WVD becomes

$$W_s(t, \omega) = W(t, \omega) + \frac{1}{2!}\left\{ W_{tt}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t'^2 G(t', \omega')dt'd\omega' + W_{\omega\omega}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2 G(t', \omega')dt'd\omega' \right\} + \cdots. \quad (11)$$

Then the bias error due to the smoothing is

$$\begin{aligned} E_s(t, \omega) &= W_s(t, \omega) - W(t, \omega) \\ &= \frac{1}{2!}\left\{ W_{tt}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t'^2 G(t', \omega')dt'd\omega' + W_{\omega\omega}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2 G(t', \omega')dt'd\omega' \right\} \\ &+ \frac{1}{4!}\left\{ W_{tttt}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t'^4 G(t', \omega')dt'd\omega' + 6W_{tt\omega\omega}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t'^2\omega^2 G(t', \omega')dt'd\omega' \right. \\ &\left. + W_{\omega\omega\omega\omega}(t, \omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega'^4 G(t', \omega')dt'd\omega' \right\} + \cdots. \end{aligned} \quad (12)$$

Eq.(12) essentially shows that the bias error is proportional to the even orders of derivatives of WVD with respect to time and frequency and the even orders of moments of window function.

As a special case, consider a Gaussian window function which can be written as

$$G(t, \omega) = \frac{1}{2\pi\,\sigma_t\,\sigma_\omega}\, e^{-\left(\frac{t^2}{2\sigma_t^2}+\frac{\omega^2}{2\sigma_\omega^2}\right)} \quad (13)$$

where, σ_t, σ_ω are standard deviations of the window along time and frequency axes respectively. The Gaussian window function is most commonly used in this two-dimensional smoothing; one of the reasons is its symmetricity in time and frequency axes.

From Eq.(13) and Eqs.(11), (12) the smoothed WVD and the bias error for a Gaussian window can be obtained as

$$W_s(t, \omega) = \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!}\left(\sigma_t^2 \frac{\partial^2}{\partial t^2} + \sigma_\omega^2 \frac{\partial^2}{\partial \omega^2} \right)^n W(t, \omega) \quad (14)$$

$$E_s(t, \omega) = \sum_{n=1}^{\infty} \frac{1}{2^n \cdot n!}\left(\sigma_t^2 \frac{\partial^2}{\partial t^2} + \sigma_\omega^2 \frac{\partial^2}{\partial \omega^2} \right)^n W(t, \omega). \quad (15)$$

If n=1 WVD can be approximated by employing up to the 2nd order in Taylor series expansion, then

$$E_s(t, \omega) \approx \frac{1}{2}\left\{ \sigma_t^2 W_{tt}(t, \omega) + \sigma_\omega^2 W_{\omega\omega}(t, \omega) \right\}. \quad (16)$$

From Eq.(16) one can observe that if W(t, ω) is concave, $W_{tt}(t, \omega)$ and $W_{\omega\omega}(t, \omega)$ are positive, then the error will be positive. This means that if WVD is spiky downwards at an arbitrary time and frequency, then the smoothing process levels up the WVD at that point. As a similar consideration, if WVD is convex at a fixed time and frequency, the smoothing levels down the WVD.

Fig.2 shows the result of a computer simulation of a pure sine signal of 100 Hz frequency. The A of Fig.2 represents WVD. The value of WVD increases with time in the first half of the record, and decreases in the second half. This is because the window length varies with time as mentioned before, therefore the larger the window length, the greater the energy within the window. Henceforth, if the window length becomes larger, the value of WVD will be larger as shown. The B of Fig.2 is the smoothed WVD using a Gaussian window function as a smoothing window. Because of the smoothing effect, WVD has the same values for the entire time axis at the frequency of 100Hz. The C represents only the difference between the B and the A, and the D is the error using Eq.(16) applied to the A. The shape of the C and the D are resemblant, but the values are quite different, and the difference becomes larger as time goes to the center of the record, where WVD is most spiky. This discrepancy occurs because WVD is spiky at those times and frequencies where discontinuity in WVD takes place and the assumption of continuity in the derivation of Eq. (16) is violated. As mentioned, although Eq.(16) is not well applicable to the spiky WVD, it can show the shape of error even for this extreme case.

For a signal in which WVD does not have spikes, that is WVD is continuous at all times and frequencies, applicability of Eq.(16) is also tested. A special signal which has continuous WVD is prepared, and the simulation result for that signal is illustrated at Fig.3. The A of Fig.3 is the WVD without smoothing, and is quite more continuous than that of Fig.2, therefore the error using Eq.(16) is very similar to the true error, not only in terms of shape but also in terms of absolute value. The only significant discrepancy in the C and the D occurs where the WVD suddenly increases, in other words, where there is discontinuity. It is noteworthy that WVD is concave at that time and frequency, so the error is positive. At other times and frequencies, the C and the D have similar values, especially at the time and frequency where WVD has the smallest negative value, the absolute values of the C and the D are not much different.

From the above simulations, one can conclude that the derived result concerning the bias error due to smoothing, is well applicable to the signal of which WVD has no discontinuity, but it has some limited features when WVD has discontinuity, although it can show the trend of the error.

Next is to check the appropriateness of Eq.(12) which expresses the bias error due to smoothing window functions other than Gaussian window. The rectangular and Hann windows are selected for this purpose. It is noteworthy that rectangular and Hann windows are expressed in terms of shapes with respect to their lengths, on the other hand, a Gaussian window is determined by its standard deviation. For the systematic comparison, it is therefore necessary to relate the lengths of rectangular and Hann windows to the standard deviation of a Gaussian window. The relation can be accomplished by imposing equal energies within all the windows; the lengths of rectangular and Hann windows are equivalent to 2.5 and 5 times respectively, of the standard deviation of a Gaussian window.

From the above relations, and having the energy is conserved during the smoothing; scale the windows of which energy is to be unity, the bias error, considering up to second order term, due to rectangular and Hann windows can be readily obtained as

$$E_R(t, \omega) = 0.26 \left\{ \sigma_t^2 W_{tt}(t, \omega) + \sigma_\omega^2 W_{\omega\omega}(t, \omega) \right\} \quad (17)$$

$$E_H(t, \omega) = 0.41 \left\{ \sigma_t^2 W_{tt}(t, \omega) + \sigma_\omega^2 W_{\omega\omega}(t, \omega) \right\} \quad (18)$$

where, $E_R(t, \omega)$ and $E_H(t, \omega)$ denote the bias errors due to rectangular and Hann windows respectively, and σ_t and σ_ω are the standard deviations of equivalent Gaussian window as written by Eq. (16).

From Eqs.(16), (17), and (18), one can predict that the bias error will be smallest for the case of using a rectangular window, and will be largest if one uses a Gaussian window.

Fig.4 depicts the smoothed WVD using a rectangular window of 100Hz sine signal which is the same as that used in Fig.2, and the bias error which is the difference between the true WVD and the smoothed WVD. Fig.5 is the results for the same signal, but using a Hann window as the smoothing window. All the smoothed WVDs and the bias errors are similar for the cases of using rectangular, Hann, and Gaussian windows(Fig.2), but smoothed WVD by using a rectangular window shows some fluctuation, because of side lobe effects.

The maximum absolute error and RMS value of the error due to each window are shown in Table.1. Bias error due to a rectangular window has the smallest value, and the error due to a Gaussian window has the largest value, as anticipated before. Moreover, as one can see from Fig.4, it is noteworthy that the smaller bias error does not promise better, smoothed WVD; small bias error means less smoothing effect, and large bias error indicates larger smoothing effect. This indicates that bias error and smoothing effect has a trade-off.

Table 1. Bias error due to various smoothing window.

Window Type	Max. Error	RMS of Error
Rectangular	1.982631	0.1434711
Hann	1.983360	0.1435499
Gaussian	1.983940	0.1436095

4. CONCLUSIONS

To see the effects of finite record on the estimation of WVD in practice, a window which has time varying length is examined. Its length increases linearly with time in the first half of the record, and decreases from the center of the record. The bias error due to this window decreases inversely proportionally to the window length and the resolution in frequency domain increases proportionally to the window length as time increases in the first half. In the second half, he bias error increases and the resolution decreases as time increases.

The bias error due to the smoothing of WVD, which is obtained by two-dimensional convolution of the true WVD and the smoothing window, which has fixed lengths along time and frequency axes, is derived for arbitrary smoothing window function. In the case of using a Gaussian window as a smoothing window, the bias error is found to be expressed as an infinite summation of differential operators. It is demonstrated that the derived formula is well applicable to the continuous WVD, but when WVD has some discontinuities, it shows the trend of the error. This is a consequence of the assumption of the derivation, that is the continuity of WVD. For windows other than Gaussian window, the derived equation is shown to be well applicable for the prediction of the bias error.

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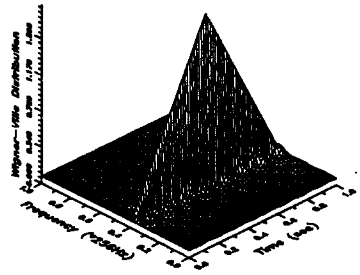
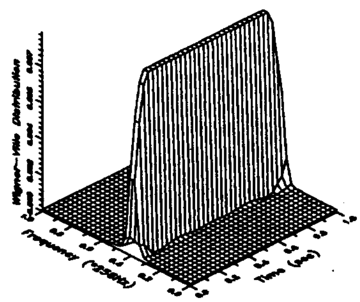
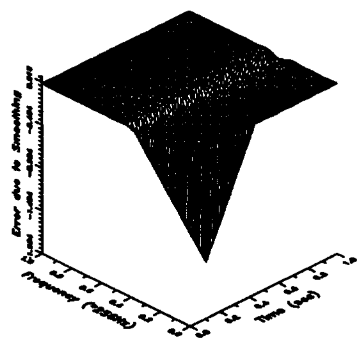


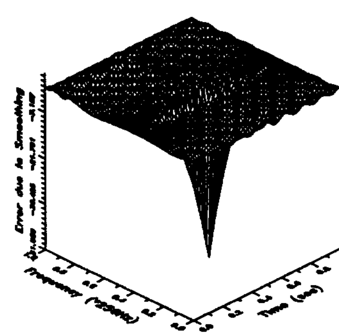
A. WVD without smoothing



B. Smoothed WVD

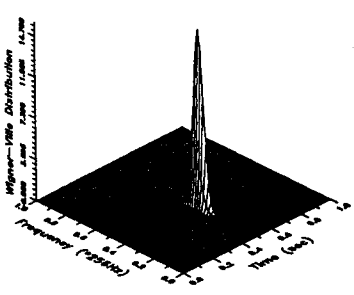


C. Error : B - A

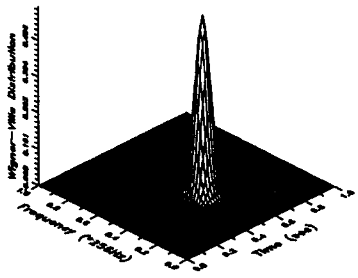


D. Error : Analytic result

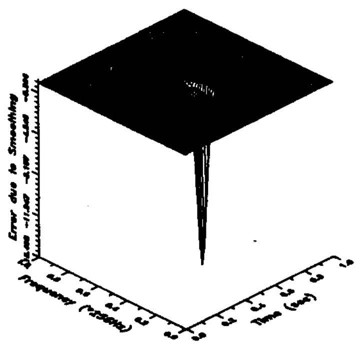
Fig.2 WVD and the bias error due to smoothing using Gaussian window function for the signal of pure sine of which frequency is 100Hz
(Number of data = 512, Sampling frequency = 512Hz, $\sigma_t = 20\Delta t$, $\sigma_\omega = 20\Delta\omega$).



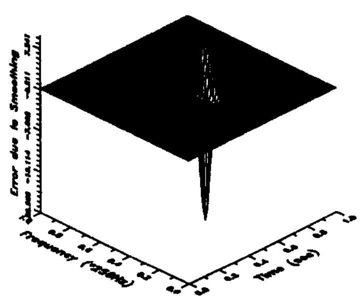
A. WVD without smoothing



B. Smoothed WVD

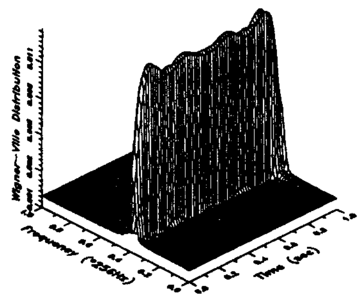


C. Error : B - A

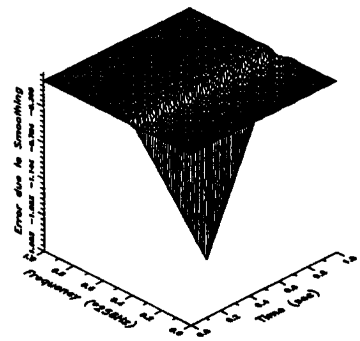


D. Error : Analytic result

Fig.3 WVD and the bias error due to smoothing using Gaussian window functio for the signal which has continuous WVD (Number of data = 512, Sampling frequency = 512Hz, $\sigma_t = 20\Delta t$, $\sigma_\omega = 20\Delta\omega$).

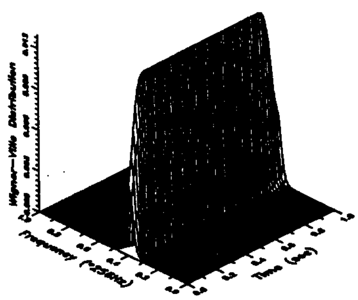


A. Smoothed WVD

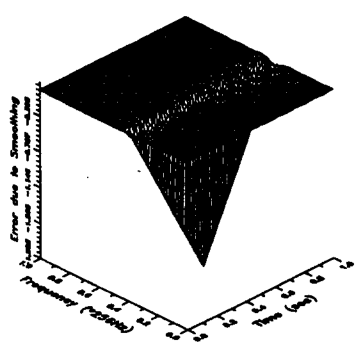


B. True bias error

Fig.4 Smoothed WVD and the bias error due to smoothing using rectangular window for the signal used in Fig.2.



A. Smoothed WVD



B. True bias error

Fig.5 Smoothed WVD and the bias error due to smoothing using Hann window for the signal used in Fig.2.