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Non-Linear Torsional Oscillations of a System incorporating a Hooke's Joint

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1 Introduction

In this paper, the problem of non-linear torsional oscillation of a system incorporating a Hooke's joint is studied. Classical perturbation methods including higher order averaging and bifurcation theory are adopted for analysis.

The equation of motion derived by Porter [1] is presented and the type of the system is identified. It has been found that two important cases deserve extensive study. Method of higher order averaging which is a main research tool in this study is introduced briefly. The averaged equations are studied analytically and numerically and the method of averaging has been found to be effective to study complex non-linear system.

2 Formulation of the problem

As shown in figure 1, the system under investigation consists of a rigid body of moment of inertia I, which is driven through a Hooke's joint and two torsionally flexible shafts. The shafts are uniform along the length and the torsional stiffnesses of the input and output shafts are s_1 and s_2, respectively. The input angular velocity, Ω , is kept to be constant. It is assumed that the source of energy loss of the system is the viscous damping, c, exerted to the rotating body, I. The angular misalignment, λ of the two shafts, is assumed to be small and this assumption makes it possible to consider this system as a weakly non-linear system.

For the Hooke's joint, the well-known relation between the input, output angles and misalignment is

$$\tan\alpha = \cos\lambda \tan\beta \, , \tag{1}$$

from which it can be shown that

$$\frac{\beta'}{\alpha'} = \frac{\cos\lambda}{1-\sin^2\lambda\cos^2\alpha} \, , \tag{2}$$

where the primes denote differentiation with respect to t. If it is assumed that no energy is dissipated at the Hooke's joint,

$$s_2(\beta-\theta)\beta' = s_1(\Omega t-\alpha)\alpha' \, . \tag{3}$$

If the angle of rotation of the rigid body, I, is θ, its equation of motion can be written as

$$I\theta'' + c\theta' = s_2(\beta-\theta) \, . \tag{4}$$

Through the same procedure as in [1], an equation of motion of the relative torsional displacement, i.e., the twist of the input shaft can be obtained from equations (2), (3) and (4) as follows:

$$\begin{aligned} \ddot{x} &+ K^2 x + 2\nu K(\dot{x}+\rho) \\ &+ \epsilon\left[\left\{\left(\rho-\frac{3}{2}\right)\cos(2x+2\tau)-\frac{3}{2}\right\}\ddot{x}\right. \\ &\quad -\rho(\dot{x}+1)^2\sin(2x+2\tau) \\ &+ \left\{(1-\rho)\ddot{x}\sin(2x+2\tau)+2(1-\rho)(\dot{x}+1)^2\cos(2x+2\tau)\right. \\ &\quad \left. -\frac{3}{2}K^2(1+\cos(2x+2\tau))\right\}x \\ &\quad \left. +2(1-\rho)\dot{x}(\dot{x}+1)\sin(2x+2\tau)\right] = 0, \end{aligned} \tag{5}$$

where x is the relative torsional displacement of the input shaft, K is the ratio of linear natural frequency ω, and input angular velocity, Ω, ρ is the ratio of the output shaft stiffness, s_1, and the stiffness of the whole linear system, s_1+s_2, and the dot represents differentiation with respect to τ, which is equal to Ωt. ϵ has been set to $\sin^2\lambda$, and ν is $\frac{c}{2\omega I}$. When there is no angular misalignment, i.e., when $\epsilon=\sin^2\lambda=0$, equation (5) reduces to the equation of motion for damped free oscillation, and the twist of input shaft, x, will diminish gradually as expected.

To identify the type of the system (5) more easily, we expand the trigonometric functions in equation (5) in Taylor series and convert it into the form $\ddot{x}+f(x,\dot{x},t)=0$. Then it can be written as

$$\begin{aligned} \ddot{x} &+ K^2x+2\nu K(\dot{x}+\rho) \\ &+ \epsilon(-\rho\sin(2\tau)-4x\rho\cos(2\tau)+2\dot{x}\sin(2\tau) \\ &\quad +2\dot{x}^2\sin(2\tau)+6\rho x^2\sin(2\tau) \\ &\quad -2x^3K^2\cos(2\tau)-8\rho x\dot{x}^2\cos(2\tau)\cdots) = 0 \end{aligned} \tag{6}$$

As shown in equation (6), there are linear and non-linear parametric excitations and external excitations. Since the frequency of the parametric excitations is 2, parametric resonances can occur for the cases, $K\approx 1,2,3,4,\cdots$ [3]. Among them, the most important cases, $K\approx 1$ and $K\approx 2$ are chosen for this study. When $K\approx 1$, the parametric primary resonance will play an important role in the response of the system, but when $K\approx 2$, the resonance due to external excitation as well as parametric secondary resonance will give significant contribution to the response of the system. It is also noted that the stiffness ratio, ρ, acts as the amplitudes of external and parametric excitations.

Before analytical and numerical works are presented, a representative non-stationary response of the original system, equation (5), is shown in figure 2. The response has been obtained by direct-time integration of equation (5) with gradually increasing or decreasing the angular velocity Ω of the input shaft. It clearly shows that large torsional vibrations occur around $K\approx 1$ and $K\approx 2$.

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3 Analysis

3.1 Bifurcation set for parametric instability

General parametrically excited system has been known to have finite responses within some regions in parameter space [3]. The bifurcation sets or Strutt diagrams which give the criteria for the onset of the finite response can be obtained by utilizing Floquet theory [3, 4]. Since the Floquet theory describes the behavior of linear ordinary differential equations in the form $\ddot{x} + f(t)x + g(t)\dot{x} = 0$, equations (5) or (6) is linearized and non-homogeneous terms including external excitations are removed from the linearized equation of motion. Then the linear ordinary differential equation is in the desired form. A variety of methods including Hill's determinant, method of strained parameters, method of multiple time scales and method of averaging etc. can be applied to the linear ordinary differential equation with periodic coefficients to obtain the bifurcation sets [4]. In this work, the method of strained parameters has been adopted to obtain explicit expressions for the bifurcation sets.

When $K \approx 1$, the variable and parameter in equation (6) are rescaled as:

$$x = \epsilon z, \nu = \epsilon\mu. \tag{7}$$

Then equation (6) is converted to

$$\ddot{z} + K^2 z + 2\mu K\rho - \rho\sin(2\tau) + O(\epsilon) = 0. \tag{8}$$

Via a transformation,

$$z = y + \frac{\rho}{K^2 - 2^2}\sin 2\tau - \frac{2\mu\rho}{K}, \tag{9}$$

equation (8) becomes

$$\ddot{y} + K^2 y + O(\epsilon) = 0, \tag{10}$$

where $O(\epsilon)$ terms include external excitations and constant terms. After the external excitations and the constant terms are removed, the method of strained parameters can be applied to give the bifurcation set as follows:

$$\begin{aligned} K^2 &= 1^2 \pm \frac{1}{2}\epsilon\left(\rho^2 - 16\mu^2\right)^{\frac{1}{2}} \\ &+ \epsilon^2\left\{\frac{1}{32}\left(7\rho^2 + 56\rho - 32 - 64\mu^2\right) \pm \frac{3}{4}\left(\rho^2 - 16\mu^2\right)^{\frac{1}{2}}\right\} \\ &+ O\left(\epsilon^3\right) \end{aligned} \tag{11}$$

When $K \approx 2$, the variable and parameter in equation (6) are rescaled as:

$$x = \epsilon z, \nu = \epsilon^2\mu. \tag{12}$$

Then equation (6) is converted to

$$\ddot{z} + K^2 z - \rho\sin(2\tau) + O(\epsilon) = 0. \tag{13}$$

Again the external excitations including $-\rho\sin 2\tau$ and constant terms are removed. Now the method of strained parameters can be applied to give the bifurcation set as

$$\begin{aligned} K^2 &= 2^2 \\ &+ \epsilon^2\left[\frac{1}{6}\left(\rho^2 + 18\rho - 6\right) \pm \frac{1}{4}\left\{\left(3\rho^2 - 6\rho - 2\right)^2 - 1024\mu^2\right\}^{\frac{1}{2}}\right] \\ &+ O\left(\epsilon^3\right) \end{aligned} \tag{14}$$

Removing external excitations and constant terms simply from equations (10) and (13) to obtain linear ordinary differential equations in the form $\ddot{x} + f(\tau)\dot{x} + g(\tau)x = 0$ might be too crude approximation, but the result can be tested by doing direct-time integration of the original equation of motion (5).

The bifurcation sets obtained from equations (11) and (14) are shown in figures 3 and 4. In figure 3, it can be found that the parametric instability is more probable for $K \approx 1$ than for $K \approx 2$ just like general parametrically excited system. In figure 4, the bifurcation sets for $K \approx 1$ are plotted in $O(\epsilon), O(\epsilon^2)$ and $O(\epsilon^3)$. The symbols o and × are the results of direct-time integration of the origianal equation of motion (5) and denote the existence and diminishing of the steady-state finite parametric response, respectively. It shows that the higher order bifurcation sets give more accurate criteria for the onset of parametric resonance. Preliminary result of direct-time integration of the original equation (5) for the case $K \approx 2$ shows that there is no qualitative change in the response when the values of the parameters are varied across the bifurcation set.

In the following it will be shown that method of averaging can be used to get the bifurcation sets for the parametric instabilities and that, although in different form, the graphic representations of them are very close to those obtained by the method of strained parameters.

3.2 Higher order averaging

The method of averaging [5, 6] has been an important research tool for the study of weakly non-linear system. Although the theory has been developed to general nth order averaging, it has been applied mainly to the first order and rarely to the second order [7, 8]. In this study, third order averaging has been found necessary to catch non-linear characteristic of the system and the procedure for the third order averaging is explained briefly in the following. For more general explanation, the readeres are referred to Murdock [9] and Guckenheimer and Holmes [10].

The method of averaging in first order consists of replacing a system in 2π-periodic standard form

$$\dot{x} = \epsilon f(x, t, \epsilon), \tag{15}$$

by an autonomous system

$$\dot{z} = \epsilon\bar{f}(z), \tag{16}$$

where

$$\bar{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z, t, 0)\, dt. \tag{17}$$

The procedure consists of two steps. The first step is a near-identity transformation such that

$$x = y + \epsilon u_1(y, t) + \epsilon^2 u_2(y, t) + \epsilon^3 u_3(y, t) + O\left(\epsilon^4\right) \tag{18}$$

where u_i is 2π-periodic in t. Expanding equation (15) in powers of ϵ gives

$$\dot{x} = \epsilon f_1(x, t) + \epsilon^2 f_2(x, t) + \epsilon^3 f_3(x, t) + \epsilon^4 \hat{f}(x, t, \epsilon) \tag{19}$$

Via the near-identity transformation (equation (18)), equation (19) becomes

$$\dot{y} = \epsilon g_1(y, t) + \epsilon^2 g_2(y, t) + \epsilon^3 g_3(y, t) + \epsilon^4 \hat{g}(y, t, \epsilon) \tag{20}$$

Differentiating equation (18) with respect to t gives

$$\begin{aligned} \dot{x} &= \dot{y} + \epsilon\left(\frac{\partial u_1}{\partial y}\dot{y} + \frac{\partial u_1}{\partial t}\right) + \epsilon^2\left(\frac{\partial u_2}{\partial y}\dot{y} + \frac{\partial u_2}{\partial t}\right) \\ &\quad + \epsilon^3\left(\frac{\partial u_3}{\partial y}\dot{y} + \frac{\partial u_3}{\partial t}\right) + O\left(\epsilon^4\right) \end{aligned} \tag{21}$$

Substituting equation (20) into equation (21) gives

$$\begin{aligned} \dot{x} &= \epsilon\left(g_1 + \frac{\partial u_1}{\partial t}\right) + \epsilon^2\left(g_2 + \frac{\partial u_1}{\partial y}g_1 + \frac{\partial u_2}{\partial t}\right) \\ &\quad + \epsilon^3\left(g_3 + \frac{\partial u_1}{\partial y}g_2 + \frac{\partial u_2}{\partial y}g_1 + \frac{\partial u_3}{\partial t}\right) + O\left(\epsilon^4\right) \end{aligned} \tag{22}$$

Substituting equation (18) into equation (19) gives

$$\dot{x} = \epsilon f_1 + \epsilon^2(f_1' u_1 + f_2) + \epsilon^3(f_1' u_2 + \frac{1}{2} f_1'' u_1^2 + f_2' u_1 + f_3) + O(\epsilon^4) \quad (23)$$

By equating coefficients of the same powers of ϵ in equations (22) and (23), we can obtain following relations for $g_1(y,t), g_2(y,t)$ and $g_3(y,t)$ in equation (20):

$$\begin{aligned} g_1(y,t) &= f_1 - \frac{\partial u_1}{\partial t}, \\ g_2(y,t) &= f_2 + f_1' u_1 - \frac{\partial u_1}{\partial y} g_1 - \frac{\partial u_2}{\partial t}, \quad (24) \\ g_3(y,t) &= f_3 + f_1' u_2 + \frac{1}{2} f_1'' u_1^2 + f_2' u_1 - \frac{\partial u_1}{\partial y} g_2 - \frac{\partial u_2}{\partial y} g_1 - \frac{\partial u_3}{\partial t} \end{aligned}$$

Now u_i's are chosen so that $g_i(y,t) \equiv g_i(y)$, i.e., g_i's are independent of time. From the first of equation (24),

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= f_1(y,t) - g_1(y,t) \\ &\equiv f_1(y,t) - g_1(y). \end{aligned} \quad (25)$$

Thus

$$u_1(t) = \int_0^t f_1(y,s) - g_1(y)\, ds + C_1(y) \quad (26)$$

and the 2π-periodicity of $u_1(t)$ gives

$$g_1(y) = \frac{1}{2\pi} \int_0^{2\pi} f_1(y,s)\, ds, \quad (27)$$

which indicates that $g_1(y)$ is the average of $f_1(y,t)$. Similarly it can be shown that

$$\begin{aligned} g_2(y) &= \frac{1}{2\pi} \int_0^{2\pi} f_2 + f_1' u_1 - \frac{\partial u_1}{\partial y} g_1\, ds, \\ g_3(y) &= \frac{1}{2\pi} \int_0^{2\pi} f_3 + f_1' u_2 \\ &\quad + \frac{1}{2} f_1'' u_1^2 + f_2' u_1 - \frac{\partial u_1}{\partial y} g_2 - \frac{\partial u_2}{\partial y} g_1\, ds. \end{aligned} \quad (28)$$

Therefore the transformed equation (20) can be rewritten as follows:

$$\dot{y} = \epsilon g_1(y) + \epsilon^2 g_2(y) + \epsilon^3 g_3(y) + \epsilon^4 \hat{g}(y,t,\epsilon), \quad (29)$$

which is autonomous up to third order.

3.3 Averaged equation when $K \approx 1$.

The variable and parameters in equation (6) are rescaled with the small parameter ϵ, which is the angular misalignment, as follows:

$$\begin{aligned} x &= \epsilon z, \\ \nu &= \epsilon \mu, \end{aligned} \quad (30)$$

and a detuning parameter is introduced as

$$K^2 = 1^2 + \epsilon\sigma. \quad (31)$$

Then equation (6) becomes

$$\ddot{z} + z - \rho \sin(2\tau) + 2\mu\rho + \epsilon F_1(z,\tau) + \epsilon^2 F_2(z,\tau) + \epsilon^3 F_3(z,\tau) + O(\epsilon^4) = 0, \quad (32)$$

where F_i's include various linear and non-linear terms.

By letting $x_1 = z, x_2 = \dot{z}$, we can express equation (32) in vector form as follows:

$$\dot{X} = AX + Q_e + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + O(\epsilon^4), \quad (33)$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, Q_e = \begin{pmatrix} 0 \\ \rho\sin(2\tau) - 2\mu\rho \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 \\ -F_1 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ -F_2 \end{pmatrix}, f_3 = \begin{pmatrix} 0 \\ -F_3 \end{pmatrix}.$$

Let

$$X = e^{A\tau} Y + \bar{Q}_e, \quad (34)$$

where

$$\begin{aligned} \bar{Q}_e &= \begin{pmatrix} -\frac{1}{3}\rho\sin(2\tau) - 2\mu\rho \\ -\frac{2}{3}\rho\cos(2\tau) \end{pmatrix}, \\ Y &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \end{aligned}$$

Transforming equation (33) via equation (34) gives the equation in the standard form (equation (15) or equation (19)) as follows:

$$\dot{y} = \epsilon g_1(y,\tau) + \epsilon^2 g_2(y,\tau) + \epsilon^3 g_3(y,\tau) + O(\epsilon^4) \quad (35)$$

where $g_i(y,\tau) = e^{-A\tau} f_i(y,\tau)$. By applying equations (27), (28) and (29) to equation (35), we can obtain an averaged equation up to third order as follows:

$$\dot{y} = \epsilon g_1(y) + \epsilon^2 g_2(y) + \epsilon^3 g_3(y) \quad (36)$$

Or

$$\begin{aligned} \dot{y}_1 &= A_1 y_1 + A_2 y_2 + A_3 y_1^2 + A_4 y_2^2 + A_5 y_1 y_2 + A_6 y_2^3 + A_7, \\ \dot{y}_2 &= B_1 y_1 + B_2 y_2 + B_3 y_1^2 + B_4 y_2^2 + B_5 y_1 y_2 + B_6 y_1^3 + B_7, \end{aligned} \quad (37)$$

where the coefficients A_i's and B_i's are functions of the parameters, ϵ, μ and ρ and the lengthy expressions are not presented here. Among them, $A_3, A_4, A_5, A_6, B_3, B_4, B_5$ and B_6 are the coefficients for the quadratic and cubic non-linearities and can be obtained by averaging up to the third order.

Representative bifurcation diagrams of equations (37) are shown in figure 5. At lower damping, multiple stable solutions co-exist and initial conditions will decide the final response of the system. Two unstable saddle type equillibrium solutions are also shown in the bifurcation diagrams. As shown in the phase plot (figure 5c), they are important in deciding basin boundaries.

As stated earlier, the method of averaging can be applied to get the bifurcation sets for the parametric instabilities. Equation (10), which is in the form $\ddot{y} + f(\tau)y + g(\tau)\dot{y} = 0$ has $y = 0$ as a solution. Then the averaged equations of the linear ordinary differential equation with periodic coefficients will be in the form

$$\begin{aligned} \dot{y}_1 &= A_1 y_1 + A_2 y_2, \\ \dot{y}_2 &= B_1 y_1 + B_2 y_2. \end{aligned} \quad (38)$$

The condition for the trivial solution $y = y_1 = y_2 = 0$ to change the stability is $A_1 B_2 - A_2 B_1 = 0$, which is also the bifurcation set for the parametric primary instability. Although its expression is in the complicated implicit form $f(K,\epsilon,\rho,\mu) = 0$ or $f(\sigma,\epsilon,\rho,\mu) = 0$, it has been shown that its graphical representatin is very close to that obtained by the method of strained parameters.

3.4 Averaged equation when $K \approx 2$.

The variable and parameters in equation (6) are rescaled with the small parameter ϵ, which is the angular misalignment, as follows:

$$\begin{aligned} x &= \epsilon z, \\ \nu &= \epsilon^2 \mu, \\ \rho &= \epsilon \delta, \end{aligned} \quad (39)$$

It is noted that ν and ρ are rescaled in different order from that for the case, $K \approx 1$, and a detuning parameter is introduced as

$$K^2 = 2^2 + \epsilon\sigma. \quad (40)$$

Then equation (6) becomes

$$\ddot{z} + 2^2 z + \epsilon F_1(z,\tau) + \epsilon^2 F_2(z,\tau) + \epsilon^3 F_3(z,\tau) + O(\epsilon^4) = 0, \quad (41)$$

where F_i's include various linear and non-linear terms.

By letting $x_1 = z, x_2 = \dot{z}$, we can express equation (41) in vector form as follows:

$$\dot{X} = AX + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + O(\epsilon^4), \tag{42}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{bmatrix} 0 & 1 \\ -2^2 & 0 \end{bmatrix},$$

$$f_1 = \begin{pmatrix} 0 \\ -F_1 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ -F_2 \end{pmatrix}, f_3 = \begin{pmatrix} 0 \\ -F_3 \end{pmatrix}.$$

Let

$$X = e^{A\tau} Y, \tag{43}$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Transforming equation (42) via equation (43) gives the equation in the standard form (equation (15) or equation (19)) as follows:

$$\dot{y} = \epsilon g_1(y, \tau) + \epsilon^2 g_2(y, \tau) + \epsilon^3 g_3(y, \tau) + O(\epsilon^4) \tag{44}$$

where $g_i(y, \tau) = e^{-A\tau} f_i(y, \tau)$. By applying equations (27), (28) and (29) to equation (44), we can obtain an averaged equation up to third order as follows:

$$\dot{y} = \epsilon g_1(y) + \epsilon^2 g_2(y) + \epsilon^3 g_3(y) \tag{45}$$

Or

$$\begin{aligned} \dot{y}_1 &= A_1 y_1 + A_2 y_2 + A_3 y_1^2 + A_4 y_2^2 + A_5, \\ \dot{y}_2 &= B_1 y_1 + B_2 y_2 + B_3 y_1 y_2 + B_4, \end{aligned} \tag{46}$$

where the coefficients A_i's and B_i's are also functions of the parameters, ϵ, μ and ρ and the lengthy expressions are not presented here. Among them, A_3, A_4 and B_3 are the coefficients for the quadratic non-linearities and those can be obtained by averaging up to the third order.

Representative bifurcation diagrams of equations (46) are shown in figure 6. At lower damping, multiple stable solutions co-exist and initial conditions will decide the final response of the system. Similarly as in the case, $K \approx 1$, two unstable saddle type equillibrium solutions are shown in the bifurcation diagrams. As shown in the phase plot (figure 6b), they are important in deciding basin boundaries.

4 Conclusions and Recommendations

The original complex non-linear system has been found to have the parametric excitations as well as the external excitations. The parametric primary resonance is important for the case $K \approx 1$ and the external resonance is also important for the case $K \approx 2$. By the use of the method of strained parameters, the bifurcation sets for the parametric resonances have been obtained and compared with those obtained by the method of averaging.

The method of averaging has been found to be an effective tool to study the complex non-linear system to get the bifurcation sets and the bifurcation diagrams of the system.

While in numerical experiment for the original equation (5), beating or modulated responses have been found and in figure 7 is shown representative time series, a phase plot and a corresponding Poincarè section. These seemingly unsteady-state responses are found for large values of the perturbation parameters, ϵ. For the large values of ϵ, the averaged equations don't seem to approximate the original equation successfully. Therefore a method for strongly non-linear system is necessary for the study of these periodic steady-state response.

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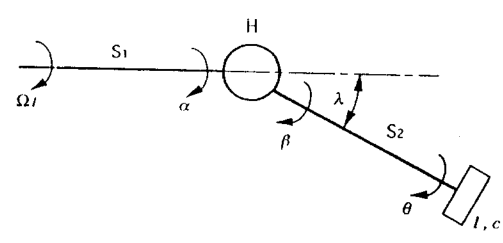


Figure 1: Diagram of the system.

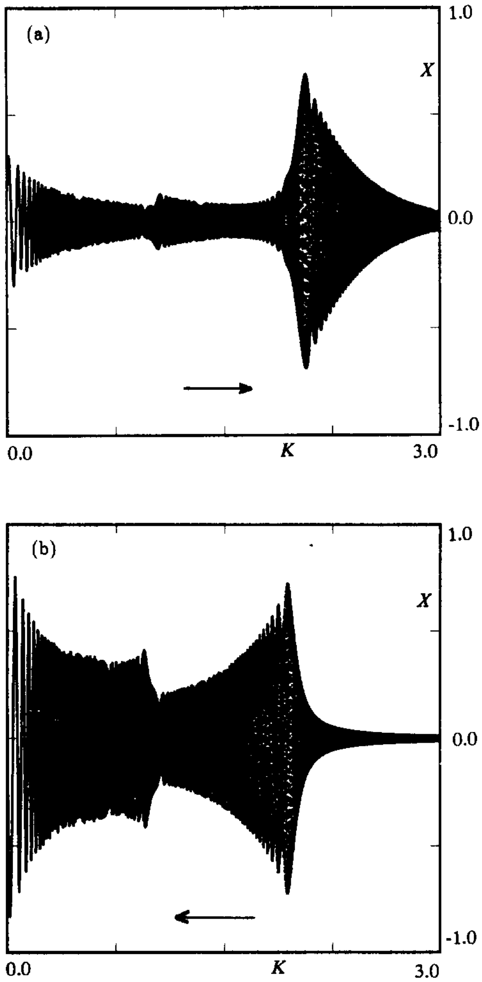


Figure 2: Non-stationary responses of the original system, equation (17) for $\nu = 0.001, \rho = 0.5, \epsilon = 0.1$ and $\dot{K} = \pm 0.001$, (a) decreasing Ω (increasing K), (b) increasing Ω (decreasing K).

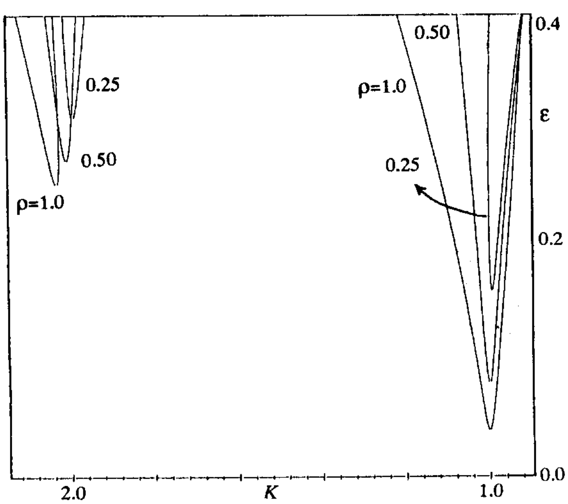


Figure 3: Bifurcation sets for $\nu = 0.01$ and $\rho = 0.25, 0.50$ and 1.0.

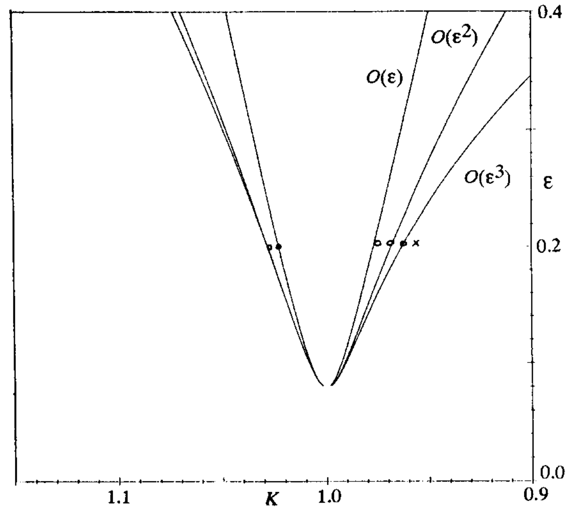
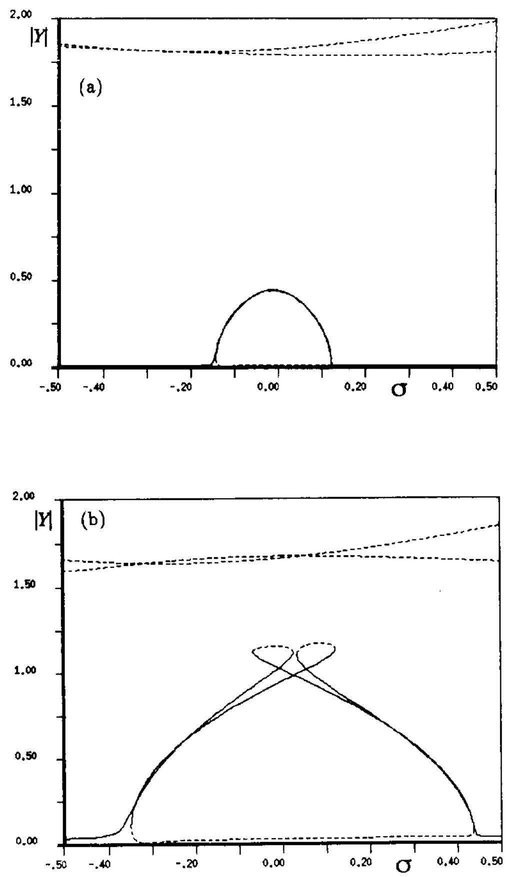


Figure 4: Bifurcation sets for $\nu = 0.01$ and $\rho = 0.50$ in $O(\epsilon), O(\epsilon^2)$ and $O(\epsilon^3)$.



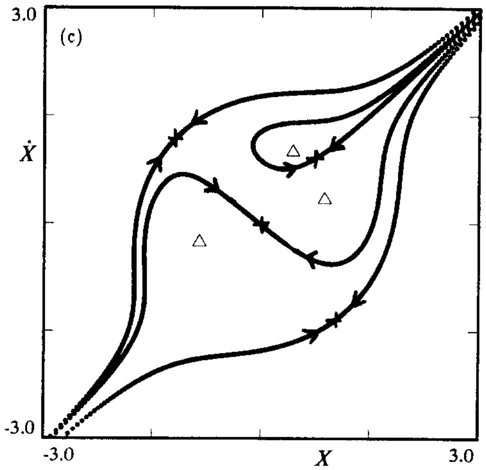


Figure 5: For $K \approx 1$, (a) bifurcation diagrams for $\nu = 0.01, \rho = 0.6, \epsilon = 0.09$, (b) bifurcation diagrams for $\nu = 0.01, \rho = 0.6, \epsilon = 0.19$, (c) phase plot for $\nu = 0.01, \rho = 0.6, \epsilon = 0.19, \sigma = 0.1$.

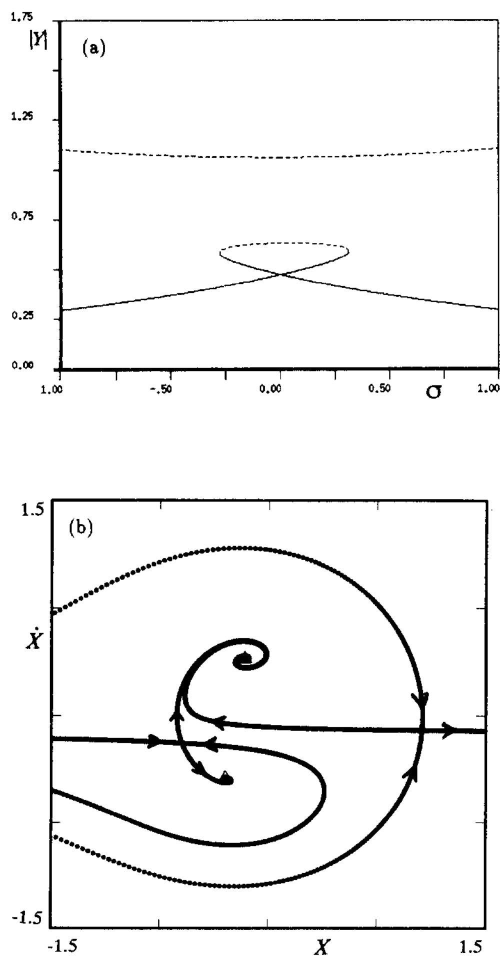


Figure 6: For $K \approx 2$, (a) bifurcation diagrams for $\nu = 2.0 \times 10^{-4}, \rho = 0.5, \epsilon = 5.0 \times 10^{-3}$, (b) phase plot for $\nu = 2.0 \times 10^{-4}, \rho = 0.5, \epsilon = 5.0 \times 10^{-3}, \sigma = 0.2$.

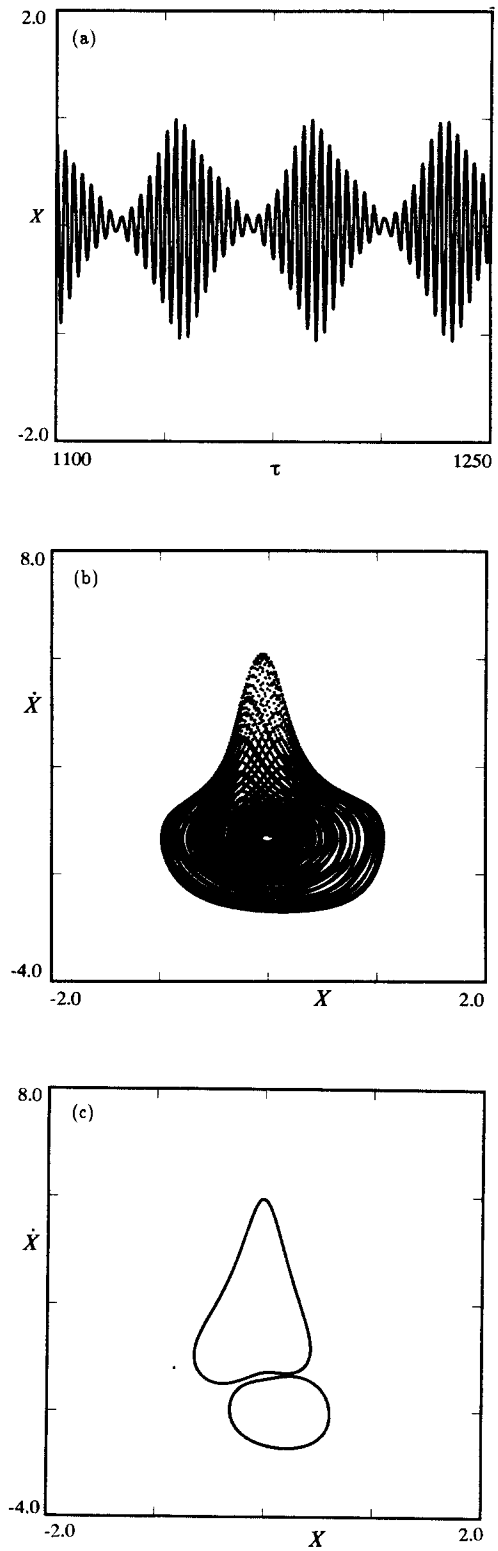


Figure 7: Periodic response for $\nu = 0.01, \rho = 0.2, \epsilon = 0.33, K = 1.98$, (a) X vs. τ, (b) phase plot in $X - \dot{X}$, (c) a corresponding Poincarè section.