Exact Poisson Distribution in the Use of NPB with Aiming Errors

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Abstract

A neutral particle beam(NPB) aimed at the object and receive a small number of neutron signals at the detector to estimate the mass of an object. Since there is uncertainty about the location of the axis of the beam relative to the object, we could have aiming errors which may lead to incorrect information about the object. Under the two assumptions that neutral particle scattering distribution and aiming errors have a circular normal distribution respectively, we have derived an exact probability distribution of neutral particles. It becomes a Poisson-power function distribution. We proved monotone likelihood ratio property of this distribution. This property can be used to find a criteria for the hypothesis testing problem.

1. Introduction

A beam of suitable high energy neutral particles in space can be used to estimate the density of an object. If the object dimensions are known, the beam becomes a means of estimating the mass of an object (see Feller [5]). Thus the beam becomes a mass discriminator. A method of discrimination proposed here is to use a neutral particle beam(NPB) aimed at the object where the neutron signal is generated, and a small number of neutron signals are counted at the detector. Beyer and Qualls [4] showed that the return neutron particles from an object interrogation obeys Poisson statistics.

The mean neutron signal λ for the Poisson distribution is computed by the *bistatic radar formula*:

$$\lambda = [I\tau] \cdot \left[\frac{A_t}{\pi(R\sqrt{2}\sigma_1)^2} \right] \cdot K(E,\theta) \cdot \left[\frac{A\varepsilon}{4\pi r^2} \right] \tag{1}$$

where I is the probe current in amperes divided by 1.602×10^{-19} coulombs, τ is the dwell time in seconds, A_t is the object area in m^2, R is the probe to object distance

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in meters, $\sqrt{2}\sigma_1$ is the beam half divergence angle, $K(E,\theta)$ is the mean number of neutrons leaked from the object per incident particle, E is the probe particle energy in election volts, θ is the scattering angle, A is the detector area in m^2, ε is the detector efficiency, and r is the object to detector distance in meters. Note that $K(E,\theta)$ depends on the mass of the object. A detailed description of this formula is given in the report of the American Physical Society report [1].

Let

$$S = \frac{A\varepsilon\tau I}{(Rr\sigma_1)^2} \quad \text{and} \quad f(E,\theta) = \frac{K(E,\theta)A_t}{4\pi} \tag{2}$$

where $f(E,\theta)$ combines parameters specific to the object design. Note that the mean signal in (1) becomes $\lambda = Sf/(2\pi)$.

2. Exact Neutron Counts Distribution in the Presence of Aiming Errors

We assume that the return neutron particles from an object interrogation during the given dwell time obeys Poisson statistics. The interrogation requires the true value of the parameters to compute the mean of the Poisson statistics. One source of errors in measurement is aiming errors (or tracking and pointing errors) which is the uncertainty about the location of the axis of the beam relative to the object. Wehner [8] studied the aiming error distribution of NPB. In this paper we consider aiming errors of the beam for an object interrogation and make the following two assumptions about aiming errors.

(i) The beam has a circular Gaussian distribution of intensity with standard deviation σ_1. This distribution is on a plane perpendicular to the beam axis.

(ii) Aiming errors yield a circular Gaussian distribution of the beam axis relative to the object center. The standard deviation of the distribution is σ_2.

Beckman and Johnson [3] give evidence from an experiment that assumption (i), that the beam has a circular Gaussian distribution of intensity, is not correct. They argue that the beam has a Pearson Type Ⅶ distribution of intensity. This distribution is much heavier in the tails than is the Gaussian.

We wish to calculate the probability that exactly x neutron particles, $x = 0, 1, 2, \dots$, are received by the single detector in presence of aiming errors. In this case the mean return signal λ is

$$\lambda = (2\pi)^{-1} Sf(E,\theta)\, e^{-(\omega_1^2+\omega_2^2)/(2\sigma_1^2)} \tag{3}$$

where S and f are defined in (2) and σ_1 is a standard deviation of the circular Gaussian intensity distribution of the beam at the object, and (ω_1, ω_2) are coördinates of points on beam cross section.

The probability of exactly x neutron particles being counted, under the assumption of a Poisson distribution of counts and aiming errors, is then

$$P(x|\lambda) = \frac{1}{x!} \int_{\omega_2=-\infty}^{\infty} \int_{\omega_1=-\infty}^{\infty} e^{-\lambda} \lambda^x \, e^{-(\omega_1^2+\omega_2^2)/(2\sigma_2^2)} \, \frac{d\omega_1 d\omega_2}{2\pi \sigma_2^2} \tag{4}$$

where λ is defined in (3) and σ_2 is a standard deviation of the circular Gaussian aiming error distribution of the beam relative to the object. The factor $2\pi \sigma_2^2$ in the denominator of the integrand of (4) is the normalizing factor of the (ω_1, ω_2) distribution, and it comes from

$$\int_{\omega_2=-\infty}^{\infty} \int_{\omega_1=-\infty}^{\infty} e^{-(\omega_1^2+\omega_2^2)/(2\sigma_2^2)} \, d\omega_1 d\omega_2 = 2\pi \sigma_2^2,$$

where we made successive changes of variables $\omega_1 = r\cos\theta$, $\omega_2 = r\sin\theta$, and $y = r^2$. We average over the aiming error distribution in (4) to modify discrimination for this uncertainty. In repeated sequential interrogation, (4) leads to a reasonable and correct modification.

Put

$$\lambda = k e^{-(\omega_1^2+\omega_2^2)/(2\sigma_1^2)} \tag{5}$$

where $k = (2\pi)^{-1} S f(E,\theta)$ and it represents the mean return neutron counts without aiming errors. Now substituting (5) into (4) and changing to polar coördinates, we obtain

$$P(x|\lambda) = \frac{1}{x!} \int_0^{\infty} e^{-k e^{-\frac{r^2}{2\sigma_1^2}}} k^x e^{-\frac{x r^2}{2\sigma_1^2}} e^{-\frac{r^2}{2\sigma_2^2}} \frac{r dr}{\sigma_2^2} \tag{6}$$

Letting

$$t = k e^{-\frac{r^2}{2\sigma_1^2}} \quad \text{and} \quad \ell = \left(\frac{\sigma_1}{\sigma_2}\right)^2, \tag{7}$$

we obtain

$$P(x|\lambda) = \frac{\ell}{k^{\ell} x!} \gamma(x+\ell\,;k), \tag{8}$$

where

$$\gamma(\nu;k) = \int_0^k t^{\nu-1} e^{-t} \, dt$$

is the incomplete gamma function.

We have defined in (5) that k be the mean number of return neutron signals counted with the assumption that no aiming errors are made in the measurement of the parameters and that the beam is perfectly centered on the object. Consider the probability distribution in (8) by $P(x;k,\ell)$

$$
\begin{aligned}
P(x;k,\ell) &= \frac{\ell}{k^{\ell}x!}\int_0^k e^{-\omega}\,\omega^{x+\ell-1}\,d\omega \\
&= \frac{1}{x!}\int_0^k e^{-\omega}\,\omega^{x}\,\ell\,(\frac{1}{k})^{\ell}\,\omega^{\ell-1}\,d\omega \\
&= \frac{1}{x!}E_{\omega}(e^{-\omega}\omega^{x})
\end{aligned}
$$

where E_{ω} represents expected value of ω, and ω has a probability distribution

$$
f(\omega) = \ell k^{-\ell}\omega^{\ell-1}, \qquad \ell > 1,\ 0 \le \omega \le k \tag{9}
$$

The distribution in (9) is called the power-function distribution. From the above expression, the distribution we have derived is a special case of a *compound Poisson distribution* where $\omega = ke^{-r^2/2\sigma_1^2}$ has a power-function distribution, and ω is a mean of the Poisson distribution. See Johnson and Kotz [6] for the definition of compound Poisson distribution. Thus the probability distribution represented by (8) may be reasonably called a *Poisson-power function distribution.*

The mean and variance of the Poisson-power function random variable, by an elementary reasoning often used in Bayesian statistics, are

$$
E(X) = E_{\omega}(E(X|\omega)) = E_{\omega}(\omega) = k\left(\frac{\ell}{\ell+1}\right)
$$

and

$$
\begin{aligned}
Var(X) &= E_{\omega}[Var(X|\omega)] + Var_{\omega}[E(X|\omega)] \\
&= E_{\omega}(\omega) + Var_{\omega}(\omega) \\
&= k\left(\frac{\ell}{\ell+1}\right) + k^2\left\{\frac{\ell}{\ell+2} - \left(\frac{\ell}{\ell+1}\right)^2\right\}.
\end{aligned}
$$

By using the same reasoning, we can find the moment generating function corresponding to the Poisson-power function distribution. That is

$$
M_X(t) = E(e^{tX}) = E(E(e^{tX}|\omega)) = E_{\omega}(e^{\omega(e^t-1)}), \tag{10}
$$

which gives

$$
M_X(t) = M((\ell,\ \ell+1,\ k(e^t-1)),
$$

where M is a Kummer's function (see Abramowitz [2] p.504) and is defined by

$$
M(a,b,x) = 1 + \frac{ax}{b} + \frac{(a)_2x^2}{(b)_2 2!} + \cdots + \frac{(a)_n x^n}{(b)_n n!} + \cdots
$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, and $(a)_0 = 1$. From the moment generating function the moments of any order can easily be evaluated by the usual differentiation procedure. Kim [7] studied and proved some properties such as unimodality, stochastical ordering, computational formula, of this distribution.

3. Monotone Likelihood Ratio Property

In this section we will show that the Poisson-power distribution has a monotone likelihood ratio. First we need lemmas that are properties of the incomplete gamma function.

Lemma 3.1. Let γ be the incomplete gamma function defined by

$$\gamma(\nu;k) = \int_0^k t^{\nu-1} e^{-t} dt.$$

Then $\gamma(\nu+a+b;t)\,\gamma(\nu;t) - \gamma(\nu+a;t)\,\gamma(\nu+b;t) > 0$ for all $\nu>0$, $t>0$, $a>0$, and $b>0$.

Proof. Writing this quantity as an integral, we have

$$\begin{aligned}
&\int_0^t \int_0^t (x^{\nu+a+b-1} y^{\nu-1} - x^{\nu+a-1} y^{\nu+b-1}) e^{-x} e^{-y} dxdy \\
&= \int_0^t \int_0^t x^a(x^b-y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy \\
&= \int\int_{(0<y<x<t)} x^a(x^b-y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy \\
&\qquad + \int\int_{(0<x<y<t)} x^a(x^b-y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy \\
&= \int\int_{(0<y<x<t)} x^a(x^b-y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy \\
&\qquad - \int\int_{(0<y<x<t)} y^a(x^b-y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy \\
&= \int\int_{(0<y<x<t)} (x^a-y^a)\,(x^b-y^b)\,(xy)^{\nu-1} e^{-(x+y)} dxdy > 0.
\end{aligned}$$

Lemma 3.2. The function $f(t;\nu) = \nu e^t t^{-\nu} \gamma(\nu;t)$ is monotone decreasing from e^t to 1 as $0 \le \nu \to \infty$ and monotone increasing from 1 to ∞ as $0 \le t \to \infty$.

Proof. First write, by a change of variable, that

$$f(t;\nu) = \int_0^1 \nu z^{\nu-1} e^{t(1-z)} dz. \tag{11}$$

From (11) it is clear that f is increasing in t and that $\lim_{t\to 0} f(t;\nu) = 1$. Also one finds by computation that

$$\lim_{t\to\infty} f(t;\nu) = \nu\cdot\Gamma(\nu) \lim_{t\to\infty} t^{-\nu} e^t = \infty.$$

Second, by an integration by parts in (11), one obtains

$$f(t;\nu) = 1 + t\int_0^1 z^\nu e^{t(1-z)} dz. \tag{12}$$

Because z in (12) is less than 1, it is clear that f is decreasing in ν, and by the dominated convergence theorem, that the limit is 1 as $\nu\to\infty$ and is e^t as $\nu\to 0$.

Lemma 3.3. The function $g(t,\nu) = \gamma(\nu+1;t)/\gamma(\nu;t)$ is monotone increasing from 0 to t as $0 \leq \nu \to \infty$, and monotone increasing from 0 to ν as $0 \leq t \to \infty$.

Proof. First to show the monotone property of g in ν, consider the ratio

$$\frac{g(t,\nu+\varepsilon)}{g(t,\nu)} = \frac{\gamma(\nu+\varepsilon+1;t)}{\gamma(\nu+\varepsilon;t)} \frac{\gamma(\nu;t)}{\gamma(\nu+1;t)} .$$

The ratio is greater than 1 by Lemma 3.1. So g is monotone decreasing in ν. Now compute

$$\lim_{\nu\to\infty} g(t,\nu) = \lim_{\nu\to\infty} \frac{\frac{1}{\nu+1} e^{-t} t^{\nu+1} f(t,\nu+1)}{\frac{1}{\nu} e^{-t} t^{\nu} f(t,\nu)}$$

where function f is defined in Lemma 3.2. Then by Lemma 4.2,

$$\lim_{\nu\to\infty} g(t,\nu) = \lim_{\nu\to\infty}\left(\frac{\nu}{\nu+1}\right) t \left(\frac{1+t\int_0^1 z^{\nu+1} e^{t(1-z)}dz}{1+t\int_0^1 z^{\nu} e^{t(1-z)}dz}\right) = t,$$

and the limit as $\nu \to 0$ is 0. Second, to show the monotone property of g in t, consider the derivative of g with respect to t

$$g'(t,\nu) = \frac{t^{\nu} e^{-t}\gamma(\nu;t) - \gamma(\nu+1;t)\, t^{\nu-1} e^{t}}{\gamma^2(\nu;t)} .$$

Then

$$\frac{\gamma^2(\nu;t)\, g'(t,\nu)}{t^{\nu-1} e^{-t}} = t\gamma(\nu;t) - \gamma(\nu+1;t)$$
$$= \int_0^t (t-y)\, y^{\nu-1} e^{-y} dy > 0 .$$

Thus g is monotone increasing in t. Now

$$\lim_{t\to\infty} g(t,\nu) = \frac{\Gamma(\nu+1)}{\Gamma(\nu)} = \frac{\nu\Gamma(\nu)}{\Gamma(\nu)} = \nu$$

and

$$\lim_{t\to 0} g(t,\nu) = \lim_{t\to 0} \frac{\int_0^t e^{-x} x^{\nu} dx}{\int_0^t e^{-x} x^{\nu-1} dx} = \lim_{t\to 0} \frac{t^{\nu+1}\int_0^1 e^{-tz} z^{\nu} dz}{t^{z}\int_0^1 e^{-tz} z^{\nu} dz} = 0 .$$

Theorem 3.1 The Poisson-power function distribution in (8) has a monotone likelihood ratio; and the Neyman-Pearson test for the hypotheses of $H_0 : k = t$ vs. $H_1 : k = d$, when $d < t$, is a left-tail test.

Proof. From (8) the likelihood ratio of the Poisson-power function distribution is

$$L(x) = \frac{P(x;d,\ell)}{P(x;t,\ell)} = \left(\frac{d}{t}\right)^{-\ell} \frac{\gamma(x+\ell;d)}{\gamma(x+\ell;t)} .$$

Now by Lemma 3.3, for $d<t$

$$\frac{L(x+1)}{L(x)} = \frac{\gamma(\nu+1;d)}{\gamma(\nu;d)} \Big/ \frac{\gamma(\nu+1;t)}{\gamma(\nu;t)}$$
$$= \frac{g(d,\nu)}{g(t,\nu)} < 1 .$$

Thus $L(x)$ is monotone decreasing function of x. It implies that the Neyman-Pearson test for an object interrogation with aiming errors is a left-tail test.

References

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