A TWO-DIMENSIONAL MAXIMUM SEARCH METHOD BY A GLOBAL PRIORITY STRATEGY WITH LOCAL PEAK ESTIMATION: ITS OPTIMAL SWITCHING CRITERION

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Abstracts The paper presents a new global maximum search method for multimodal unknown functions of two variables. The search method is composed of two stages and sequentially samples the candidate point in a subdomain selected using a priority function in each stage. The search domain is auto-similarly divided into triangular subdomains, or cells, during the search process. A measure of accuracy of local maximum search is introduced to check if a local search has converged to a specified accuracy or the maximum of a local peak cannot be the global maximum. A criterion for switching from the first to the second stage, is proposed using a ratio of the observed peak width to the largest cell in the domain. By numerical simulations, the required number of trials is evaluated for some function models with different peak parameters, and the switching criterion is optimally determined. The results show that the proposed method obtains global maximum points with certainty and saves largely computation time even for functions with extremely steep peaks.

Keywords Two-dimensional maximum search, Global priority strategy, Local peak estimation, Optimal switching criterion, Numerical simulation

1. INTRODUCTION

Optimization problems in a wide area of control engineering mostly require efficient and robust algorithms for searching global maxima of multimodal unknown functions of several intricate variables. Up to the present, some search methods using new control techniques such as simulated annealing[4], neural networks[2], and genetic algorithms[1] have been presented. However, there are some problems in practical use of these methods, in particular due to unsatisfactory reliability and efficiency.

Sequential search has the possibility of performing better decisions concerning the succeeding trials by utilizing information acquired from previous ones. Owing to the great complexity of search procedure for functions of more than two variables, no effective methodology or integrated strategy combining global and local search is established.

This paper describes an extension of the previously proposed maximum search method using a combined global priority strategy with local peak estimation[5] to two-dimensional multimodal unknown functions. The two-dimensional search method utilizes a simplified arrangement algorithm, where a search domain is represented as a set of isosceles right-angled triangular subdomains, or cells, with three adjoining sampled points and auto-similarly divided into triangular cells during the search process.

In this paper, a criterion for deciding the local maxima as possible global maxima or switching from the first to the second stage, is proposed using a ratio of the observed peak width to the largest cell as a direct extension of the one-dimensional case. It is shown that, owing to the auto-similar triangular division of cells and the criterion for local maximum estimation, the combined strategy holds down the total number of trials to the minimum and gives a reasonable combination of global and local search. The optimal switching criterion is determined through computer simulations for some example functions with different peak parameters.

2. TWO-DIMENSIONAL SEARCH ALGORITHM

2.1 Basic structure of the algorithm

We describe the basic structure of the proposed two-dimensional maximum search algorithm. For a set of i sampled points $\{\mathbf{x}_1, y_1; \ldots; \mathbf{x}_i, y_i\}$ of an unknown continuous function $y = f(\mathbf{x})$ defined over $\mathbf{x} \in \Omega = [0,1]^2$, the search domain is represented as a set of auto-similarly divided triangular cells, whose vertices are already sampled points. The k-th cell is defined as a 4-tuple: $\Gamma_k = (\gamma_{k1}, \gamma_{k2}, \gamma_{k3}, \gamma_{k4})$ $(k = 1, \ldots, k_0)$, where k_0 denotes the total number of cells in the search domain by the i-th trial, γ_{k1}, γ_{k2}, and γ_{k3} denote serial numbers of vertices of the cell, and γ_{k4} denotes the number of the longest edge of the serially numbered edges $\gamma_{k1}\gamma_{k2}$, $\gamma_{k2}\gamma_{k3}$, and $\gamma_{k3}\gamma_{k4}$.

The candidate point in the k-th cell for the next trial is denoted by $\mathbf{x}_k^+$, and is defined as the midpoint of the longest edge. The distance between the candidate point and the farthest already sampled points in the cell is called the radius of the unsearched area and denoted by d_k.

The maximum search algorithm is composed of two stages. In the first stage, starting from the several regularly sampled points, candidate points are evaluated using a priority function and the next sampled point is selected. If the values of the radius d_k are equal for two cells, the observed values at the two end points are also considered. In the second stage, where local maxima are discovered using globally sampled points, the local peaks are preferentially searched. After the new trial, cells in the search domain are consistently reconfigured so that the algorithm can be efficiently iterated.

The different priority functions are considered for the first stage, where information concerning the shape of the unknown function is almost scarce, and for the second stage, where local peaks can be estimated using the observed values of sampled points. The functions are defined as follows:

(1) for the first stage,

$$p_k = \{1\} \cdot d_k, \quad (1)$$

(2) for the second stage,

$$p_k = \{1 + (K_p - 1)Y_k\} \cdot d_k, \quad (2)$$

where

$Y_k = (y_k^+ - y_{min})/(y_{max} - y_{min})$

$y_k^+ = \max\{y(\gamma_{k1}), y(\gamma_{k2}), y(\gamma_{k3})\}$

$y_{max} = \max\{y_1, \ldots, y_i\}$

$y_{min} = \min\{y_1, \ldots, y_i\}$.

Y_k is a canonical variable such that if $y_k^+ = y_{max}$ then $Y_k = 1$ and $p_k = K_p \cdot d_k$, and if $y_k^+ = y_{min}$ then $Y_k = 0$ and $p_k = 1 \cdot d_k$.

Thus, the search algorithm is summarized as follows.

(1) $i = 5$, and arrange the initial four cells, Γ_1, Γ_2, Γ_3, and Γ_4.

(2) Select $\mathbf{x}_{i+1} = \mathbf{x}_{k^*}$, where $k^* = \max p_k$.

(3) Observe the value of the sampled point $y_{i+1} = f(\mathbf{x}_{i+1})$.

(4) Update cells.

Step 1:

if $\gamma_{k^*4} = 1$ then $k_0 = k_0 + 1$, $\Gamma_{k^*} = (\gamma_{k^*1}, i+1, \gamma_{k^*3}, 3)$ and $\Gamma_{k0} = (i+1, \gamma_{k^*2}, \gamma_{k^*3}, 2)$,

if $\gamma_{k^*4} = 2$ then $k_0 = k_0 + 1$, $\Gamma_{k^*} = (\gamma_{k^*1}, \gamma_{k^*2}, i+1, 1)$ and $\Gamma_{k0} = (\gamma_{k^*2}, i+1, \gamma_{k^*3}, 3)$,

if $\gamma_{k^*4} = 3$ then $k_0 = k_0 + 1$, $\Gamma_{k^*} = (\gamma_{k^*1}, \gamma_{k^*2}, i+1, 1)$ and $\Gamma_{k0} = (i+1, \gamma_{k^*2}, \gamma_{k^*3}, 2)$,

Step 2:

if two cells share the longest edge, update the two cells,

Step 3:

if candidate points of cells updated in Step 1 and Step 2 are already sampled points, then update the cells.

(5) Calculate the priority function for each cell.

(6) $i = i+1$, and go back to (2).

2.2 Switching of algorithm using local peak estimation

2.2.1 Measure of accuracy of local maximum search.

In this section, we introduce a measure of accuracy of maximum search for local peak estimation.

Theorem 1: Suppose a given function $f(\mathbf{x})$ is upward convex near a local peak $\mathbf{x}_1$, then the following inequality holds:

$$\frac{y_1 - y^*}{y^* - y_*} \geq \frac{\min(y_2, y_3, y_4, y_5) - y_{min}}{y_1 - y_{min}} \quad (3)$$

where

$\mathbf{x}_1$ is shared by the four cells and the maximum point of candidate points within these cells,

$\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$, and $\mathbf{x}_5$ are adjacent points,

$y_j = f(\mathbf{x}_j)$, $j=1, \ldots, 5$,

$y_{min} = f(\mathbf{x}_{min})$,

y^* is the true global maximum,

y_* is the true global minimum.

The proof is straightforward and omitted by space reason[2]. The left side term of (7) is called the accuracy of local maximum search, and the right side term gives its lowest limit. In the second stage, the priority function (1) is used in place of (2), if the lowest limit is over a reference value. Thus, it is possible to coordinate the accuracy of local search for each local peak by monitoring if the lowest limit is over a reference value.

In an advanced stage that a considearable accuracy of each peak has been attained, it is possible to determine that the maximum of a local peak cannot be the global maximum by monitoring if the lowest limit is under a reference value, which is denoted by β_0 and gives a criterion for control of preferential local search. The value of β_0 should be determined according to the class of the unknown function[6]. In this paper, we have taken $\beta_0 = 0.9$.

2.2.2 Switching criterion from the first to the second stage.

As an index representing the degree of geometrical complexity of an unknown function, the minimum peak width of all the local peaks is considered. On the other hand, as an index representing the degree of detailed search, the maximum radius, which is the maximum value of d_k for $k=1, \ldots, k_0$, is considered. Then the ratio of the minimum peak width to the maximum radius is used as a switching criterion from the first to the second stage. If the ratio is over a reference value, which is denoted by ρ_0, then the switching is performed from the first to the second stage, and vice versa.

3. SIMULATION RESULTS AND DISCUSSION

The number of local peaks and the width of the peak with the global maximum are supposed to mainly dominate the required number of trials to attain the global maximum according to a specified accuracy. In this report the required number of trials is defined as the number required till the accuracy is over a reference value or it is decided that the maxima of the local peaks cannot be the global maximum for all local peaks.

3.1 Function models

Fig. 1(a) and Fig. 2(a) show 3-D graphic surface views of the function model A and B, respectively, whose local peaks are of almost equal width and smooth near the local maximum points, with the number of local peaks m = 8.

The equation of the function model A is defined as

$$f(x_1, x_2) = \sum_{j}^{m}(5 + 5h_j)\exp[-60m\{(x_1 - r_{1j})^2 + (x_2 - r_{2j})^2\}]. \quad (4)$$

In Eq. (4), the coordinates r_{1j} and r_{2j} of each local peak and h_j are randomly decided.

The equation of the function model B is defined as

$$f(x_1, x_2) = \max_j [(5 + 5h_j)[1 - \sqrt{16/m\{(x_1 - r_{1j})^2 + (x_2 - r_{2j})^2\}}], 0]. \quad (5)$$

The shape of the function model B is conic near the local maximum points, and the coordinates of each local maximum point r_{1j} and r_{2j} and h_j are the same as in case of the function model A. For both the function models A and B, the initial value of random numbers is changed such that the peak width becomes narrower as m increases. It is also changed when a distance between local maximum points is under 70 % of the standard distance (= $1/\sqrt{m}$).

Fig. 3(a) shows a 3-D graphic surface view, where the peak width of the globally maximum peak is considerably narrow in comparison with the minimum width of other local peaks.

The equation of the function model C is defined as

$$f(x_1, x_2) = 10 \exp[- 7\{(x_1 - 0.49)^2 + (x_2 - 0.51)^2\}] + 13 \exp[- 20/w\{(x_1 - 0.88)^2 + (x_2 - 0.14)^2\}] \quad (6)$$

The ratio of the narrowness is denoted by w, and w =1/8 in Fig. 3(a).

3.2 Simulation results

Fig. 1(b) shows the simulation results for the function model A, with m = 2, 4, 6, 8. The parameters of the algorithm are K_p = 100000.0, β_0 = 0.9, and μ_0 = 0.99. It is shown that the required number of trials is small for ρ_0 = 0, 0.5, 1.0, and the number is monotone increasing as ρ_0 increases over 2.0. This tendency is more clearly distinguished as the number of peaks m increases. Similar tendencies have been also confirmed for μ_0 = 0.999 and $\mu_0 \geq 0.9999$, and other initial function parameters decided randomly.

For other functions with smooth peak shape, such as Camel Back functions[3], a class of polynomial functions, similar tendencies have been obtained and the required number of trials is rather smaller for the same accuracy of local maximum search.

Fig. 2(b) and Fig. 3(b) show similar tendencies for the function model B with m = 2, 4, 6, 8, and for the function model C with w = 1/1, 1/2, 1/4, 1/8, respectively, and these tendencies have been also confirmed for m > 2.

3.3 Discussion

The main reason why the number of trials is reliably small in the case that the switching criterion ρ_0 is between 0.0 and 0.5, especially nearly 0.0, is that the number of trials required for local search can be surely decreased owing to control of preferential local search even when preferential local search has been too early initiated.

Through the computer simulations for the three function models, it has been found out that the ratio of the minimum peak width to the maximum radius when all the local peaks have been discovered is between 0.707 and 1.0. Generally, as the width of the global maximum peak becomes narrower, the ratio of the minimum peak width to the maximum radius tends to increase. However, the latter value cannot be over 1.414[5]. This fact means that the algorithm can surely discover any local peak of peak width larger than 1.414 times the maximum radius. In other words, the lower limit of the peak width of a local peak can be estimated using the value of the maximum radius. If the user has the minimum peak width to be discovered, in advance, he can stop the algorithm by monitoring the maximum radius. These characteristics are distinguished in contrast with ones brought about by other new search methods such as simulated annealing and genetic algorithms.

4. CONCLUSIONS

We have presented the two-dimensional search method by a combined global priority strategy with local peak estimation and proposed its switching criterion between the global and the preferential local search.

By computer simulations using some function models with smooth local peaks, it has been shown that the optimal value of the criterion is between 0.0 and 0.5, especially nearly 0.0, and the required number of trials is satisfactorily small with sufficiently high reliability. These results have been also confirmed even for peculiar functions with rather steep peaks and concave ridges. Furthermore, some numerical comparisons with other search methods which utilize non-triangular division of the search domain are given to show the computational advantages of the proposed method.

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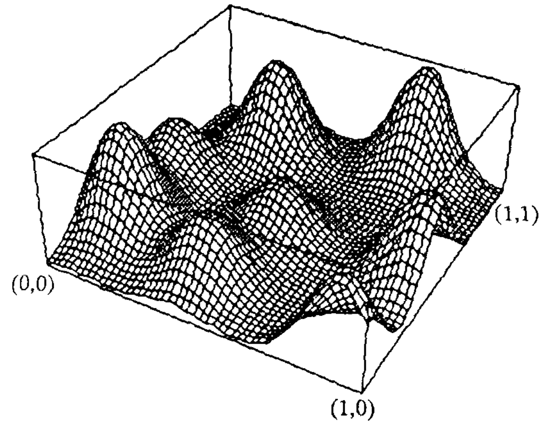

Fig. 1(a) 3-D graphic surface view of function model A (m = 8)

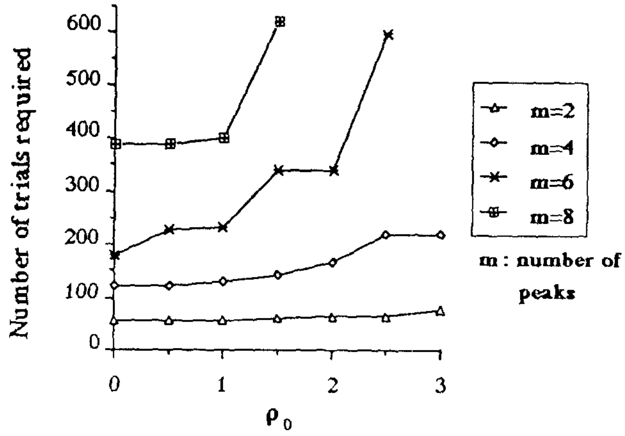

Fig. 1(b) Relations between number of trials required and switching criterion ρ_0 for function model A (K_p = 100000.0, β_0 = 0.9, and μ_0 = 0.99)

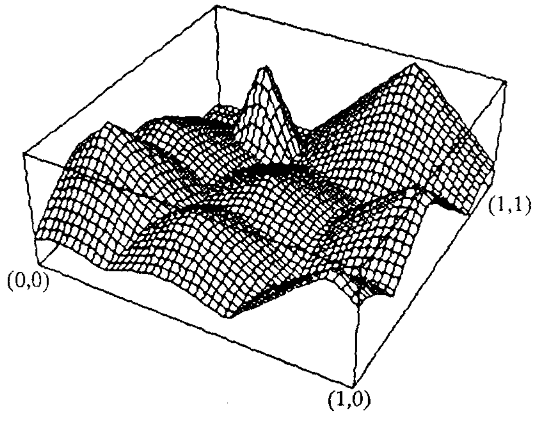

Fig. 2(a) 3-D graphic surface view of function model B (m = 8)

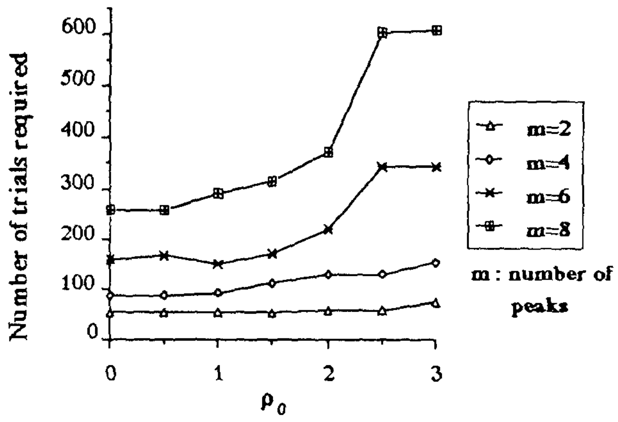

Fig. 2(b) Relations between number of trials required and switching criterion ρ_0 for function model B (K_p = 100000.0, β_0 = 0.9, and μ_0 = 0.99)

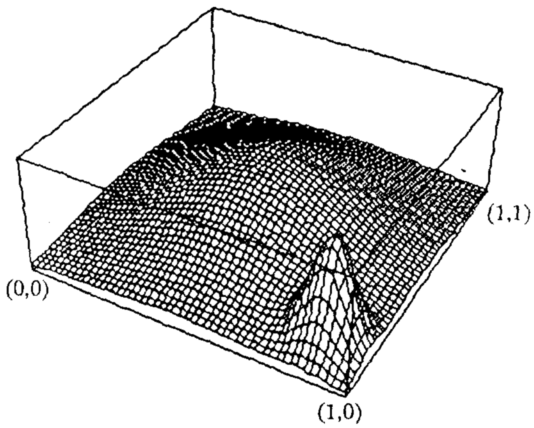

Fig. 3(a) 3-D graphic surface view of function model C (w = 1/8)

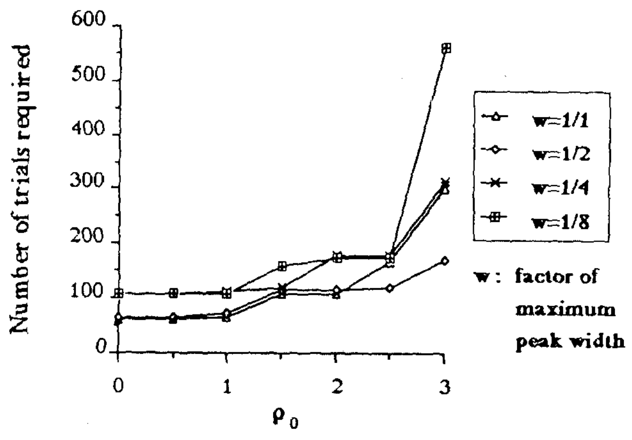

Fig. 3(b) Relations between number of trials required and switching criterion ρ_0 for function model C (K_p = 100000.0, β_0 = 0.9, and μ_0 = 0.99)