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New Method for LQG Control of Singularly Perturbed Discrete Stochastic Systems

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Abstract

In this paper a new approach to obtain the solution of the linear-quadratic Gaussian control problem for singularly perturbed discrete-time stochastic systems is proposed. The alogorithm proposed is based on exploring the previous results that the exact solution of the global discrete algebraic Riccati equations is found in terms of the reduced-order pure-slow and pure-fast nonsymmetric continuous-time algebraic Riccati equations and, in addition, the optimal global Kalman filter is decomposed into pure-slow and pure-fast local optimal filters both driven by the system measurements and the system optimal control input. It is shown that the optimal linear-quadratic Gaussian control problem for singularly perturbed linear discrete systems takes the complete decomposition and parallelism between pure-slow and pure-fast filters and controllers.

1 Introduction

Theory of singular perturbations has been very fruitful control engineering research area in the last twenty five years, [1, 2]. The discrete-time systems have been the subject of research since early eighties. Several control researchers have produced important results on different aspects of control problems of deterministic singularly perturbed discrete systems such as Phillips, Blankenship, Mahmoud, Sawan, Khorasani, Naidu and their coworkers. Particularly important are the fundamental results of Khalil and Litkouhi, [3, 4]. Along the lines of research of Khalil and Litkouhi in the papers of Gajic and Shen [5, 6] an extension of the linear-quadratic control problem of [3] and the formulation and solution of the linear-quadratic Gaussian stochastic control problem are obtained. In this paper, we introduce a completely new approach pretty much different than all other methods used so far in the theory of singular perturbations. It is well known that the main goal in the control theory of singular perturbations is to achieve the problem decomposition into slow and fast time scales. Our approach is based on a closed-loop decomposition technique which guarantees complete decomposition of the optimal filters and regulators and distribution of all required off-line and on-line computations.

In the regulation problem (optimal linear-quadratic control problem), it is shown in [7] how to decompose exactly the ill-defined discrete-time singularly perturbed algebraic Riccati equation into two reduced-order pure-slow and pure-fast well-defined continuous-time algebraic Riccati equations. Note that the reduced-order continuous-time algebraic Riccati equations are nonsymmetric, but their $O(\epsilon)$ approximations are symmetric ones. We show that the Newton method is very efficient for their solutions since the initial guesses close an $O(\epsilon)$ to the exact solutions can be easily obtained from the results already available in [3].

In the filtering problem, in addition of using duality between filter and regulator to solve the discrete-time ill-defined filter algebraic Riccati equation in terms of the reduced-order pure-slow and pure-fast well-defined continuous-time algebraic Riccati equations, we have obtained completely independent pure-slow and pure-fast Kalman filters both driven by the system measurements and the system optimal control input [7]. In the literature of linear stochastic singularly perturbed systems, it is possible to find exactly decomposed slow and fast Kalman filters ([8] for continuous-time systems, and [5] for discrete-time systems), but those filters are driven by the innovation process so that the additional communication channels have to be formed in order to construct the innovation process.

In this paper, we use the separation principle and the results of [9] and [7] to solve the linear-quadratic Gaussian control problem of singularly perturbed discrete systems. A real world control example is solved in order to demonstrate the proposed method.

2 Linear-Quadratic Gaussian Optimal Control Problem

This section presents a new approach in the study of the LQG control problem of singularly perturbed discrete systems when the performance index is defined on an infinite-time period. The discrete-time LQG problem of a singularly perturbed system has been studied for the full state feedback in [5] and for the output feedback in [10]. We will solve the LQG problem by using the results obtained in [7]. That is, the discrete algebraic Riccati equation, which is the main equation in the optimal control problem of the singularly perturbed discrete system, is completely and exactly decomposed into two reduced-order continuous-time algebraic Riccati equations.

Consider the singularly perturbed discrete linear stochastic system represented in the fast time scale by [5]

$$\begin{aligned} x_1(k+1) &= (I_{n_1}+\epsilon A_1)x_1(k)+\epsilon A_2 x_2(k) \\ &\quad +\epsilon B_1 u(k)+\epsilon G_1 w_1(k) \\ x_2(k+1) &= A_3 x_1(k)+A_4 x_2(k)+B_2 u(k) \\ &\quad +G_2 w_1(k) \qquad (1) \\ y(k) &= C_1 x_1(k)+C_2 x_2(k)+w_2(k) \end{aligned}$$

with the performance criterion

$$J=\frac{1}{2}E\left\{\sum_{k=0}^{\infty}\left[z^T(k)z(k)+u^T(k)Ru(k)\right]\right\}, \quad R>0 \qquad (2)$$

where $x_i \in R^{n_i}$, $i=1,2$, comprise slow and fast state vectors, respectively. $u \in R^m$ is the control input, $y \in R^l$ is the observed output, $w_1 \in R^r$ and $w_2 \in R^l$ are independent zero-mean stationary Gaussian mutually uncorrelated white noise processes with intensities $W_1>0$ and $W_2>0$, respectively, and $z \in R^s$ is the controlled output given by

$$z(k)=D_1 x_1(k)+D_2 x_2(k) \qquad (3)$$

All matrices are of appropriate dimensions and assumed to be constant.

The optimal control law of the system (1) with performance criterion (2) is given by [11]

$$u(k)=-F\hat{x}(k) \qquad (4)$$

with the time-invariant filter

$$\hat{x}(k+1)=A\hat{x}(k)+Bu(k)+K\left[y(k)-C\hat{x}(k)\right] \qquad (5)$$

where

$$A=\begin{bmatrix} I_{n_1}+\epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B=\begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix}$$

$$C=[C_1 \;\; C_2], \quad K=\begin{bmatrix} \epsilon K_1 \\ K_2 \end{bmatrix}$$

The regulator gain F and filter gain K are obtained from

$$F=\left(R+B^T P_R B\right)^{-1}B^T P_R A \qquad (6)$$

$$K=AP_F C^T\left(W_2+CP_F C^T\right)^{-1} \qquad (7)$$

where P_R and P_F are positive semidefinite stabilizing solutions of the discrete-time algebraic regulator and filter Riccati equations [12], respectively, given by

$$\begin{aligned} P_R &= D^T D+A^T P_R A \\ &\quad -A^T P_R B\left(R+B^T P_R B\right)^{-1}B^T P_R A \qquad (8) \end{aligned}$$

$$\begin{aligned} P_F &= AP_F A^T-AP_F C^T\left(W_2+CP_F C^T\right)^{-1}CP_F A^T \\ &\quad +GW_1 G^T \qquad (9) \end{aligned}$$

where

$$D=[D_1 \;\; D_2], \quad G=\begin{bmatrix} \epsilon G_1 \\ G_2 \end{bmatrix}$$

The required solutions P_R and P_F in the fast time scale version have the forms

$$P_R=\begin{bmatrix} P_{R1}/\epsilon & P_{R2} \\ P_{R2}^T & P_{R3} \end{bmatrix}, \quad P_F=\begin{bmatrix} \epsilon P_{F1} & \epsilon P_{F2} \\ \epsilon P_{F2}^T & P_{F3} \end{bmatrix} \qquad (10)$$

In obtaining the required solutions of (8) and (9) in terms of the reduced-order problems, [6] has used a bilinear transformation technique introduced in [13] to transform the discrete-time algebraic Riccati equation into the continuous-time algebraic Riccati equation. In our case, the exact decomposition method of the discrete algebraic regulator and filter Riccati equations produces two sets of two reduced-order nonsymmetric algebraic Riccati equations, that is, for the regulator

$$P_1 a_1-a_4 P_1-a_3+P_1 a_2 P_1=0 \qquad (11)$$

$$P_2 b_1-b_4 P_2-b_3+P_2 b_2 P_2=0 \qquad (12)$$

and for the filter

$$P_s a_{1F}-a_{4F}P_s-a_{3F}+P_s a_{2F}P_s=0 \qquad (13)$$

$$P_f b_{1F}-b_{4F}P_f-b_{3F}+P_f b_{2F}P_f=0 \qquad (14)$$

where the unknown coefficients can be obtained from the result of [7]. The Newton algorithm can be used efficiently in solving the reduced-order nonsymmetric Riccati equations (11)-(14).

It has been shown in [7] that the optimal global Kalman filter, based on the exact decomposition technique, is decomposed into pure-slow and pure-fast local optimal filters both driven by the system measurements. As a result, the coefficients of the optimal pure-slow filter are functions of the solution of the pure-slow Riccati equation only and those of the pure-fast filter are functions of the solution of the pure-fast Riccati equation only. Thus, these two filters can be implemented independently in the different time scales (slow and fast). The pure-slow and pure-fast filters are, respectively, given by

$$\begin{aligned} \hat{\eta}_s(k+1) &= \left(a_{1F}+a_{2F}P_s\right)^T\hat{\eta}_s(k) \\ &\quad +K_s y(k)+B_s u(k) \\ \hat{\eta}_f(k+1) &= \left(b_{1F}+b_{2F}P_f\right)^T\hat{\eta}_f(k) \\ &\quad +K_f y(k)+B_f u(k) \qquad (15) \end{aligned}$$

where

$$\begin{bmatrix} B_s \\ B_f \end{bmatrix}=\mathbf{T}_2^{-T}B=\left(\Pi_{1F}+\Pi_{2F}P_F\right)^{-T}B$$

Matrices Π_{1F} and Π_{2F} can be found in [7]. It should be noted that the filtering method proposed for singularly perturbed linear discrete systems allows complete decomposition and parallelism between pure-slow and pure-fast filters.

The optimal control in the new coordinates has been obtained as

$$u(k) = -F\hat{x}(k) = -F\mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s(k) \\ \hat{\eta}_f(k) \end{bmatrix}$$
$$= -\begin{bmatrix} F_s & F_f \end{bmatrix} \begin{bmatrix} \hat{\eta}_s(k) \\ \hat{\eta}_f(k) \end{bmatrix} \quad (16)$$

where F_s and F_f are obtained from

$$\begin{bmatrix} F_s & F_f \end{bmatrix} = F\mathbf{T}_2^T$$
$$= \left(R + B^T P_R B\right)^{-1} B^T P_R A \left(\Pi_{1F} + \Pi_{2F} P_F\right)^T \quad (17)$$

The optimal value of J is given by the very well-known form [11]

$$J_{opt} = \frac{1}{2} tr\left[D^T D P_F + P_R K\left(C P_F C^T + W_2\right) K^T\right] \quad (18)$$

where F, K, P_R, and P_F are obtained from (6)-(9).

Corresponding block diagram which represents clearly the proposed method is given in Figure 1. It is very interesting that the proposed scheme for the linear singularly perturbed discrete systems, with corresponding matrix coefficients, can be represented by the same structure of the block diagram as the one for the continuous systems in [14].

Example 2:

In order to demonstrate the efficiency of the proposed method, we consider a real world control system – a fifth-order discrete model of a steam power system [15]. The system matrices are given by

$$A = \begin{bmatrix} 0.9150 & 0.0510 & 0.0380 & 0.0150 & 0.0380 \\ -0.0300 & 0.8890 & -0.0005 & 0.0460 & 0.1110 \\ -0.0060 & 0.4680 & 0.2470 & 0.0140 & 0.0480 \\ -0.7150 & -0.0220 & -0.0211 & 0.2400 & -0.0240 \\ -0.1480 & -0.0030 & -0.0040 & 0.0900 & 0.0260 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0098 & 0.1220 & 0.0360 & 0.5620 & 0.1150 \end{bmatrix}^T$$

and other matrices are chosen as

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$D^T D = diag\{\ 5 \quad 5 \quad 5 \quad 5 \quad 5\ \}, \quad R = I$$

It is assumed that $G = B$ and that the white noise processes are independent and have intensities

$$W_1 = 5, \quad W_2 = diag\{\ 5 \quad 5\ \}$$

It is shown [15] that this model possesses the singularly perturbed property with $n_1 = 2$, $n_2 = 3$, and $\epsilon = 0.264$.

The obtained solutions for the LQG control problem are summarized as following.

The completely decoupled filters driven by measurements y are given as

$$\hat{\eta}_s(k+1) = \begin{bmatrix} 0.8804 & 0.0428 \\ -0.0481 & 0.7824 \end{bmatrix} \hat{\eta}_s(k) + \begin{bmatrix} 0.1045 & 0.0643 \\ 0.1717 & 0.2780 \end{bmatrix} y(k) + \begin{bmatrix} 0.0629 \\ 0.3650 \end{bmatrix} u(k)$$

$$\hat{\eta}_f(k+1) = \begin{bmatrix} 0.2606 & -0.0112 & -0.0158 \\ -0.0533 & 0.1822 & -0.0585 \\ -0.0224 & 0.0662 & 0.0069 \end{bmatrix} \hat{\eta}_f(k) + \begin{bmatrix} -0.0044 & -0.0163 \\ 0.0164 & 0.0741 \\ 0.0067 & 0.0296 \end{bmatrix} y(k) + \begin{bmatrix} -0.0458 \\ 0.5590 \\ 0.1157 \end{bmatrix} u(k)$$

The feedback control in the new coordinates is given

$$u(k) = \begin{bmatrix} 0.1407 & -0.3068 \end{bmatrix} \hat{\eta}_s(k) - \begin{bmatrix} 0.1918 & 0.3705 & 0.1019 \end{bmatrix} \hat{\eta}_f(k)$$

The difference of the performance criterion between the optimal value, J_{opt}, and the one of the proposed method, J, is given by

$$J_{opt} = 6.73495$$
$$J - J_{opt} = 0.7727 \times 10^{-13}$$

In obtaining the optimal value of performance criterion, J_{opt}, we have used a bilinear transformation technique developed in [13].

It should be noted that the results represented here in solving via the Newton method recursively the reduced-order nonsymmetric algebraic Riccati equations for filter and regulator (11)-(14), are obtained by using the number of iterations of $i = 5$, respectively.

3 Conclusion

In this paper we developed a new approach to solve the LQG optimal control for linear singularly perturbed discrete stochastic systems. The main idea of the proposed method is in the fact that the ill-defined discrete time singularly perturbed algebraic Riccati equation is exactly decomposed into two reduced-order pure-slow and pure-fast well-defined continuous time algebraic Riccati equations. A very important feature of the obtained results is that it allows complete time-scale parallelism of the filtering and control tasks through the complete and exact decomposition of the optimal control and filtering problems into slow and fast time scales, which reduces both off-line and on-line required computations.

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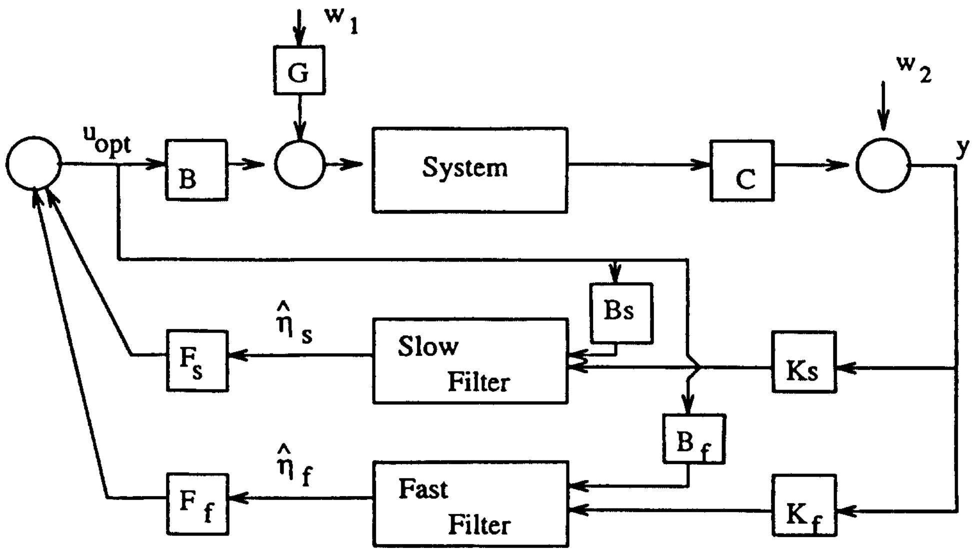

Figure 1: Block diagram representation of the reduced-order stochastic control

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