On Synthesizing Low-Order State Estimators and Low-Order H_∞ Filters

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Abstracts The standard estimation and filtering theory are well known and has recently been incorporated with the H_∞ optimization techniques where the parametrizations of all estimators and filters are utilized. The issue of reducing its order is always of interest. This paper presents a method for synthesizing low-order stable state estimators. The method presented in this paper is based on the utilization of a free parameter function contained in the parametrization of all state estimators. The results obtained in the paper are compared with standard results on low-order estimators. Both results are shown to be the same in a sense of its orders, but the approaches taken are largely different. It is also shown in the paper that the method can easily and directly be extended to the Kalman filters and the H_∞ (sub)optimal filters. Consequently, the orders of all state estimators, Kalman filters, and H_∞ filters are shown to be reduced down to the number of states minus the number of outputs, respectively.

Keywords State estimator, Low-order state estimator, Kalman filter, Low-order Kalman filter, H_∞ (sub)optimal filter, Low-order H_∞ filter.

1. Introduction

Modern control design techniques can be used to design an optimal full-state feedback design, where an estimation theory is to be incorporated.

For practical implementation, full-order estimator (Kalman filter) is usually designed first and then reduced to a low-order estimator (Kalman filter) without significantly degrading the performance and robustness [8]. The full-order estimator (Kalman filter) can be reduced through conventional model reduction techniques such as modal residualization, balanced truncated model reduction, singular perturbation approximation, and Hankel approximation. Singular perturbation approximation (also called balanced residualization) elaborates balanced truncated model reduction in a sense of discarding the poorly controllable-observable states by setting its derivative zero, not by simply truncating.

H_∞ filter is different from such state estimators in that the H_∞ filtering problem is to find an estimate of a controlled variable such that the ratio of the estimation error energy to the disturbance energy (i.e., ∞-norm) is less than a specified value. However, the problem of reducing the order of H_∞ filters may fall in the same as the one of finding low-order state estimators.

All possible state estimators, Kalman filters, and H_∞ filters are characterized respectively. A common feature in those characterizations is the existence of a free parameter function. Though the free parameter in the characterizations allows a certain degree of freedom to designers, it causes the increase in the order of the state estimators, Kalman filters, and H_∞ filters.

The method presented in this paper for synthesizing low-order state estimators (both deterministic and stochastic cases) and low-order H_∞ filters is different from the conventional techniques, but is similar in concept to [6] for low-order stabilizing controllers and [3] for low-order H_∞ suboptimal controllers. In that sense the results shown in this paper can be considered as extensions to estimation and filtering theory. The key idea is to utilize a free parameter contained commonly in the characterization of all estimators and filters, and to delete subsystems associated with the unobservable modes appeared in the full-order systems.

The paper is organized as follows. In Section 2, the characterizations of all state estimators for deterministic case and Kalman filters are summarized and then the method of reducing the orders are presented. H_∞ filtering problem is reviewed in Section 3 and it is then shown that the reduction of its order can be treated in a similar way. Conclusions follow in Section 4.

The notation adopted in this paper is fairly standard. L_2 denotes Lebesgue space of (real) rational matrices whose elements are strictly proper and have no poles in the imaginary axis. RH_∞ denotes Hardy space of real rational matrices whose elements are stable and proper. A transfer function matrix is represented in terms of state-space data by $G(s) = [A, B, C, D] = C(sI - A)^{-1}B + D$, where $A, B, C,$ and D are real matrices of appropriate dimensions.

2. State Estimation

2.1 Deterministic Case

Standard estimator theory is well known, for example [1,2]. It is further extended in [9] that if one particular state estimator is given then all possible state estimators can be generated in an affine fashion. Dual of the results for the control problem may also be found in [9]. Though freedom is given in the characterization of all state estimators, it causes the increase in the order of the state estimators. In this section, the problem of reducing the order of full-order state estimators is addressed.

The class of state estimators we consider in this section are constrained in the following way. That is, we require that the state estimate should be a ***stable*** proper linear time-invariant function of the plant input and output, and that in the absence of modelling errors and noise, and for any input signal, state estimation error should decay to zero (that is, be ***unbiased***).

We consider the nominal plant without noise

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$, and $C \in R^{p \times n}$. Let H be any matrix such that $A - HC$ is stable, then it is well known that one corresponding state estimator is given in observer form by

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}) \tag{2}$$

Then ***all*** stable, unbiased state estimators are characterized, for example in [9], and given by

$$\tilde{x} = \hat{x} + Q(y - C\hat{x}) \tag{3}$$

where $Q(s) \in RH_\infty$ is a free transfer function.

Now, by letting a state-space realization of a free parameter $Q(s)$ be

$$Q(s) = [A_q, B_q, C_q, D_q] \tag{4}$$

where $A_q \in R^{n_q \times n_q}$, $B_q \in R^{n_q \times m}$, $C_q \in R^{p \times n}$, and $D_q \in R^{p \times m}$, we obtain a state-space expression of all stable, unbiased state estimators given by

$$F(s) = [\begin{pmatrix} A - HC & 0 \\ -B_qC & A_q \end{pmatrix}, \begin{pmatrix} B & H \\ 0 & B_q \end{pmatrix}, \begin{pmatrix} I - D_qC & C_q \end{pmatrix}, \begin{pmatrix} 0 & D_q \end{pmatrix}] \tag{5}$$

It is clearly shown in the expression (5) that all state estimators $F(s)$ is stable and its order is $n + n_q$. This implies that though the freedom parameter $Q(s)$ allows the designer a certain degree of freedom on designing the state estimators, it may unnecessarily increase the order of the state estimators by n_q. The issue of reducing its order is always of interest in view of practical implementation. In the following, a method for synthesizing low-order stable state estimators is presented.

To reduce the order, we apply a state similarity transformation using a nonsingular matrix

$$T = \begin{pmatrix} I_n & 0 \\ X & I_{n_q} \end{pmatrix} \tag{6}$$

where $X \in R^{n_q \times n}$ shall be determined later on. Then the new state-space realization of all state estimators can be expressed by

$$F(s) = [\begin{pmatrix} A - HC & 0 \\ -XA + XHC - B_qC + A_qX & A_q \end{pmatrix}, \begin{pmatrix} B & H \\ -XB & -XH + B_q \end{pmatrix}, \begin{pmatrix} I - D_qC + C_qX & C_q \end{pmatrix}, \begin{pmatrix} 0 & D_q \end{pmatrix}] \tag{7}$$

So, if the following two matrix equations are satisfied for a certain matrix X,

$$\begin{aligned} A_qX - X(A - HC) &= B_qC \\ D_qC - C_qX &= I \end{aligned} \tag{8}$$

then the realization of (7) can be reduced to the following low-order realization:

$$F_r(s) = [A_q, \begin{pmatrix} -XB & -XH + B_q \end{pmatrix}, C_q, \begin{pmatrix} 0 & D_q \end{pmatrix}] \tag{9}$$

by deleting a subsystem associated with the unobservable modes.

Obviously the order of the low-order state estimators expressed by (9) is only n_q, that is much less than the full order, $n + n_q$, of the expression (7). Note that since $A - HC$ is stable, the states of the subsystem associated with $A - HC$ can be truncated without breaking the stability of the stable estimators. Note also that the low-order state estimators $F_r(s)$ in (9) is stable since A_q is stable. This means that low-order ***stable*** state estimators can always be obtained.

Two equations in (8) are similar to those appeared in [3,6], where it is proved that the solution always exists in case of $n_q = n - p$. Hence it is not difficult to show that the solution matrix X to the two equations in (8) always exists also in case of $n_q = n - p$. Here we briefly explain how to solve the two equations, leaving the interested readers to refer to the detailed procedure in [3,6]: The first equation of (8) is of a Sylvester equation type but is different from a standard Sylvester equation in that the coefficient matrices in the equation are element matrices of the free parameter function $Q(s)$. So, the first equation is solved by making use of freedom in the coefficient matrices and by a partitioning technique.; The second equation is then easily solved since it is a linear matrix equation having coefficient matrices again from element matrices of $Q(s)$.

We therefore conclude that the order of all stable state estimators can always be reduced down to $n - p$. The result obtained here may be compared with the well known standard results on reduced order estimator theory, for example [1,2], in that though the technique used is different, the order of low-order state estimators is equal to the plant state numbers minus the output numbers.

2.2 Kalman Filters

A Kalman filter can be incorporated with the LQG(linear quadratic Gaussian) control to provide an estimate of the state, in case the state variables in a stochastic system are not available.

Consider the time-invariant signal generator, having process disturbance w and measurement disturbance v:

$$\begin{aligned} \dot{x} &= Ax + Bw, x(0) = 0 \\ z &= Lx \\ y &= Cx + Dv \end{aligned} \tag{10}$$

in which $DD^T = I$ is assumed for all times of interest. The filtering problem is to find a causal, linear time-invariant filter $F(s)$ such that $\hat{z} = F(s)y$ is an optimal estimate of $z = Lx$, with L a continuous matrix valued function. Optimality here means that the 2-norm of the average RMS power of the estimation error, i.e.

$$||R||_2 = \lim_{T\to\infty} E\{\tfrac{1}{T}\int_0^T (\hat{z} - Lx)^T(\hat{z} - Lx)dt\}^{\frac{1}{2}} \tag{11}$$

is minimized, with $R : [w^T v^T]^T \rightarrow \hat{z} - z$. The Kalman filter is the optimal solution to the problem defined by (11), and the optimal filter is given in [5] by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + YC^T(y - C\hat{x}) \\ \hat{z} &= L\hat{x} \end{aligned} \tag{12}$$

in which Y is the stabilizing solution to the Riccati equation:

$$AY + YA^T - YC^TCY + BB^T = 0 \tag{13}$$

where the stabilizing solution means that $A - YC^TC$ is stable.

All stable filters with prescribed performance γ that satisfies $||R||_2 < \gamma$ are characterized in [5] by

$$\hat{z} = L\hat{x} + Q(y - C\hat{x}) \tag{14}$$

in which $\hat{x}$ is the optimal state estimate, and $Q \in RH_\infty$ with

$$||Q||_2^2 + trace(LYL^T) \leq \gamma^2 \tag{15}$$

To obtain the low-order Kalman filters, the same technique used in the previous section can be alpplied to the characterization of all full-order Kalman filters given in (14). Now, by letting a state-space realization of a free parameter $Q(s)$ be $Q(s) = [A_q, B_q, C_q, D_q]$ as in (4), a state-space expression of all stable Kalman filters is obtained by

$$F(s) = [\begin{pmatrix} A - YC^TC & 0 \\ -B_qC & A_q \end{pmatrix}, \begin{pmatrix} YC^T \\ B_q \end{pmatrix}, \begin{pmatrix} L - D_qC & C_q \end{pmatrix}, D_q] \tag{16}$$

As in the case of stable state estimators, the order of all stable Kalman filters $F(s)$ is also $n + n_q$ and thus the free parameter $Q(s)$ increases the order of the Kalman filters by n_q. So, by using the same technique as in the state estimation case, we can derive the state-space realization of low-order Kalman filters. That is, if the following two matrix equations are satisfied for a certain matrix X,

$$\begin{aligned} A_qX - X(A - YC^TC) &= B_qC \\ D_qC - C_qX &= L \end{aligned} \tag{17}$$

then the realization of (16) can be reduced to the following low-order realization:

$$F_r(s) = [A_q, -XYC^T + B_q, C_q, D_q] \tag{18}$$

which is also stable and largely depends on the element matrices of $Q(s)$.

3. H_∞ Filtering

The H_∞ filtering problem is to estimate the output z using the measurements y. The problem was considered in, for example, [4,11].

Suppose the signal is generated by the time-invariant state-space system:

$$\begin{aligned} \dot{x} &= Ax + Bw, x(0) = 0 \\ z &= Lx \\ y &= Cx + Dv \end{aligned} \tag{19}$$

in which $DD^T = I$ is assumed for all times of interest. Unlike the Kalman filtering problem, the process disturbance w and the measurement disturbance v are in L_2, i.e. unknown deterministic diturbances of finite energy. The H_∞ filtering problem is to find an estimate of $z = Lx$ of the form $\hat{z} = F(s)y$ such that $F(s)$ is stable and the ratio of the estimation error energy to the disturbance energy is less then γ^2, i.e.

$$max \frac{||\hat{z} - Lx||_2^2}{||d||_2^2} = ||R||_\infty^2 \leq \gamma^2 \tag{20}$$

for all $d = [w^T v^T]^T \in L_2$, where the mapping system $R : d \rightarrow (\hat{z} - Lx)$ is stable. We shall assume that (A, C) is detectable, and that (A, B) has no uncontrollable mode on the imaginary axis.

All H_∞ stable filters such that the system R is stable and satisfies (20) are generated in [5], in a form of lower linear fractional transformation, by

$$F(s) = F_l(F_a, Q) \tag{21}$$

where $Q \in RH_\infty$, $||Q||_\infty < \gamma$ and

$$F_a(s) = [A - Y_\infty C^TC, \begin{pmatrix} Y_\infty C^T & -\gamma^{-2}Y_\infty L^T \end{pmatrix}, \begin{pmatrix} L \\ -C \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}] \tag{22}$$

in which $Y_\infty \geq 0$ is a stabilizing solution to the algebraic Riccati equation:

$$AY_\infty + Y_\infty A^T - Y_\infty(C^TC - \gamma^{-2}L^TL)Y_\infty + BB^T = 0 \tag{23}$$

where $A - Y_\infty(C^T C - \gamma^{-2} L^T L)$ is asymptotically stable.

The results above on the H_∞ filtering problem are the analogue of the Kalman filtering results mentioned earlier. The key differences are in the Riccati equations (13) and (23) which are very similar to the covariance equations for the Kalman filtering problem with exception of the term $\gamma^{-2} Y_\infty L^T L Y_\infty$. Thus, in H_∞ filtering, the states to be estimated influence the filter itself unlike in Kalman filtering where the optimal estimate of any state-functional is obtained from the optimal state-estimator. Note that as $\gamma \to \infty$, the H_∞ filters approach the standard Kalman filters.

By letting a state-space realization of a free parameter $Q(s)$ be $Q(s) = [A_q, B_q, C_q, D_q]$ as in (4) and by using a state-space realization of the linear fractional transformation, all stable H_∞ filters $F(s) = F_l(F_a, Q)$ in (21) can be expressed in a state-space realization by

$$[\begin{pmatrix} A - Y_\infty C^T C + \gamma^{-2} Y_\infty L^T D_q C & -\gamma^{-2} Y_\infty L^T D_q \\ -\gamma^{-2} B_q Y_\infty L^T & A_q \end{pmatrix}, \begin{pmatrix} Y_\infty C^T \\ B_q \end{pmatrix}, \begin{pmatrix} Y_\infty C^T - \gamma^{-2} D_q Y_\infty L^T & C_q \end{pmatrix}, D_q] \tag{24}$$

It is also clear that the full order H_∞ filters is of $n + n_q$ and that the same technique used previously can be alpplied to the problem of reducing the order of all H_∞ stable filters. That is, the following low-order realization $F_r(s)$ of all H_∞ filters can be obtained from the realization of (24) by:

$$\begin{array}{c} F_r(s) = [A_q - \gamma^{-2} X Y_\infty L^T C_q, \\ B_q - X Y_\infty (C^T - \gamma^{-2} L^T D_q), C_q, D_q] \end{array} \tag{25}$$

if the following two matrix equations are satisfied for a certain matrix X,

$$\begin{array}{c} A_q X - X[A - Y_\infty(C^T C - \gamma^{-2} L^T L)] = B_q C \\ D_q C - C_q X = L \end{array} \tag{26}$$

Note that though the two equations in (26) can also be solved in a similar manner to the previous cases in Section 2, the stability of the low-order H_∞ filters is not automatically guaranteed, in contrast to the low-order state estimators and the low-order Kalman filters.

4. Conclusions

Problems of synthesizing low-order stable state estimators, low-order Kalman filters, and low-order H_∞ filters was considered respectively in this paper. By properly manipulating a free parameter function contained commonly in the characterizations of all state estimators, Kalman filters and H_∞ filters, it was shown that the proposed approach can be applied to all the cases considered in this paper in a unifying fashion.

The same technique adopted in this paper may be extended to the finite horizon cases for both LQG and H_∞ filters. H_2 filtering problem may also be treated in a similar way.

In practice of controller reduction, the error criterion in reducing the controller is degradation of the total performance instead of the error between the full-order controller and the reduced one, as in [7] for the LQG controller reduction and [10] for the H_∞ controller reduction. The method presented in this paper provides a systematic way to synthesizing low-order state estimators, Kalman filters, and H_∞ filters. However, the method lacks such a closed-loop consideration and thus it will be worth in future research to address this issue.

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