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Robust Moving Horizon Control of Nonlinear Systems

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Abstract: In this paper, a moving horizon control algorithm, which can be applied for a wide class of nonlinear systems with control and state constraints, is considered. In a neighborhood of the origin, a linear feedback controller is applied. Outside this neighborhood, a moving horizon control law is applied. The time taken to solve an optimal control problem is considered in the algorithm so that the proposed control law can be applied as an on-line controller.

Keywords: nonlinear systems, moving horizon control, optimal control.

1. Introduction

Moving horizon control is a promising approach to the construction of stablizing controllers for general nonlinear systems([1-10]). It is proved to be particularly useful in the process industry where the system is often subject to control and state constraints([2-9]). This type of control requires a repetitive solution of a finite-interval open loop optimal control problem. Unfortunately, the usefulness of the control is limited by the fact that the time taken to solve an optimal control problem, which is rather large, has not been considered in any moving horizon control algorithms.

In this paper, we suppose that the nonlinear system is not known exactly and consider the time taken to solve an optimal contol problem so that the proposed moving horizon control algorithm can be used as an on-line control law.

This paper is consisted that in section 2, a moving horizon control algorithm is proposed, in section 3, robust stability property is proved, and it is concluded in section 4.

2. Moving horizon control algorithm

In this paper, we consider a nonlinear system described by

$$\dot{x}(t) = f(x(t),\ u(t)) \qquad (1)$$

where $f: R^n \times R^m \rightarrow R^n$, $u(t) \in \Omega$, and Ω is a compact set in R^m. We need the following assumptions.

Assumption 2.1: (a) $f(\cdot,\cdot)$ is twice continuously differentiable and it satisfies that $f(0,\ 0) = 0$.

(b) $\Omega \in R^m$ is a compact set containing the origin in its interior. ■

Since it is not likely to know the system equation exactly, we assume that the system is modeled by

$$\dot{x}^m(t) = f^m(x^m(t),\ u(t)) \qquad (2)$$

where $f^m: R^n \times R^m \rightarrow R^n$ is twice continuously differentiable and the input $u(\cdot)$ is in the set Ω. Let us define the states of the real and the model systems at time t_0 with an initial condition x_0 and an input $u(\cdot)$ by $x(\cdot,\ t_0,\ x_0,\ u)$ and $x^m(\cdot,\ t_0,\ x_0,\ u)$, respectively. Suppose that the real system (1) and its model (2) satisfy the followings:

Assumption 2.2: (a) The model (2) satisfies that $f^m(0,\ 0) = 0$.

(b) $f(\cdot,\cdot)$ and $f^m(\cdot,\cdot)$ are Lipschitz i.e., there exists an $L \in (0,\ \infty)$ such that for any $x_1(t),\ x_2(t) \in R^n$ and $u_1(t),\ u_2(t) \in \Omega$, $t \in [\ 0,\ \infty)$ the followings are satisfied.

$$\| f(x_1(t),\ u_1(t)) - f(x_2(t),\ u_2(t)) \|_p \le L (\| x_1(t) - x_2(t) \|_p + \| u_1(t) - u_2(t) \|_p)$$

$$\| f^m(x_1(t),\ u_1(t)) - f^m(x_2(t),\ u_2(t)) \|_p \le L (\| x_1(t) - x_2(t) \|_p + \| u_1(t) - u_2(t) \|_p)$$

where $\| z \|_p$ is defined by $\langle z,\ Pz \rangle^{1/2}$ and P is an arbitrary symmetric positive definite matrix.

(c) There is a $\beta \in (0,\ \infty)$ such that for any $x(t) \in R^n$ and $u(t) \in \Omega$, $t \in [\ 0,\ \infty)$

$$\| f(x(t),\ u(t)) - f^m(x(t),\ u(t)) \|_p \le \beta (\| x(t) \|_p + \| u(t) \|_p)$$

is satisfied. ■

The original moving horizon control problem contains a constraint, which the state at the end of the horizon becomes zero. Unfortunately, with this hard constraint, it takes infinite time to solve the problem ([2,3,6]). This difficulty was solved by relaxing the constraint with that the state is in an ε-ball at the end of the horizon and in the ε-ball a linear state feedback law is applied ([8]). Even with this improvement, the control algorithm is considered as a theoretical one since the time taken to solve an optimal control problem has not been considered.

In this paper, a new moving horizon control algorithm is proposed with the consideration of the time taken to solve an optimal control problem. Let $x_i^m = x^m(t_i, t_{i-1}, x_{i-1}^m, u_{[t_{i-1}, t_i]})$ be the state of the model at time t_i. Then, the optimal control problem, $P(x_i^m)$, at the time t_i, is represented as follows.

$$P(x_i^m): \min C(x^m, t, u, T)$$

subject to

$$\psi^j(x^m, u) \le 0, \; i = 1, \cdots, l,$$

$$\| x^m(t_i + T, t_i, x_i^m, u) \|_p \le \delta/2$$

$$u \in \Omega_{[t_i, t_i+T]}$$

where T is the time interval of the moving horizon and δ is a constant that will be defined later. The solutions of the problem, $P(x_i^m)$, are the control input defined in $[t_i, t_i + T]$ $u_{[t_i, t_i+T]}$ and the time interval T_i. The constraint $\psi^j(x^m, u)$ can be a function of not only the state and the input but also the time. In general, it can be represented by $\psi^j(x^m, u) = \max_{t \in [t_i, t_i+T]} \psi^j(x^m, u, t)$, $j = 1, \cdots, l$, where $\psi^j: R^n \times R^m \times R_+ \to R$ is continuous and continuously differentiable.

With this optimal control problem, we propose the following algorithm. To simply the notations, let t_i be $i-th$ sampling time, x_i^m be the state of the model at time t_i, x_i^p be the state of the real system at time t_i and the solutions of the problem $P(x_i^m)$ be $u_i(\cdot)$ and T_i.

Moving Horizon Control Algorithm:

Data: at the initial time t_0, the initial state of the real system $x_o^p \in R^n$.

Initialization: Solve $P(x_0^p)$ to obtain the time interval T_0 and the input $u_0(t)$, $t \in [t_0, t_0 + T_0]$

Control Algorithm:

Step 0: Set $i = 0$.

Step 1: Measure the state of the real system, x_i^p.

Step 2: If $x_i^p \in B_\delta$, then apply the linear state feedback law else, go to Step 3. Here, $B_\delta = \{ x \in R^n \mid \| x \|_p < \delta \}$.

Step 3: Apply the input $u_i(t)$, $t \in [t_i, t_{i+1}]$ where $t_{i+1} = t_i + \Delta$ and Δ is the sampling period.

Step 4: Calculate the state of the model x_{i+1}^m at time t_{i+1}.

Step 5: Solve $P(x_{i+1}^m)$ to obtain the next time interval $T_{i+1} \ge \Delta$ and the control input $u_{i+1}(t)$, $t \in [t_{i+1}, t_{i+1} + T_{i+1}]$.

Step 6: Set $i = i + 1$, wait until t_{i+1} and go to Step 1. ■

Here, the sampling period Δ is chosen so that it is bigger that the time taken to solve the optimal control problem. If not, even though we wait until t_{i+1} and go to Step 1 as stated in Step 6, the control input to be applied at time t_{i+1} is not available. The region B_δ, where a linear state feedback control law is applied, will be described in the next section.

3. Robustness

In this section, the robustness property of the control law will be proved. First, we need to prove that a linear feedback control law can stablize the system if the state is near the origin. The linearized real and model systems near the origin are represented by

$$\dot{x}_L(t) = f_x(0, 0) x_L(t) + f_u(0, 0) u(t)$$

$$\dot{x}_L^m(t) = f_x^m(0, 0) x_L^m(t) + f_u^m(0, 0) u(t)$$

where $f_x(0, 0)$, $f_x^m(0, 0) \in R^{n \times n}$ and $f_u(0, 0)$, $f_u^m(0, 0) \in R^{n \times m}$ are the derivatives of $f(\cdot, \cdot)$ and $f^m(\cdot, \cdot)$ with respect to x and u, respectively. The following assumption is needed.

Assumption 3.1: The pairs of $(f_x(0, 0), f_u(0, 0))$ and $(f_x^m(0, 0), f_u^m(0, 0))$ of the linearized systems are controllable. ■

The following result can be found in [8].

Lemma 3.1: Suppose that assumptions 2.1, 2.2, and 3.1 are satisfied. Then, there is a $\delta_{\max} > 0$ and $K \in R^{m \times n}$ (or $K^m \in R^{m \times n}$) such that when for any $\delta \in (0, \delta_{\max}]$ the state $x(\tau)$ is in $B_\delta = \{ z \in R^n \mid \| z \|_p < \delta \}$ for some time τ, the linear control law $u(t) = -Kx(t) \in \Omega$ (or $u(t) = -K^m x^m(t) \in \Omega$), $t \ge \tau$, stablizes the system (1) (or (2)). ■

Since the control law has to satisfy the constraints of the optimal control problem, we need the following assumption.

Assumption 3.2: Let $\delta_{\max}$ be defined as in Lemma 3.1. Then, there exists $\delta_m \in (0, \delta_{\max})$ and K satisfying Lemma 3.1 such that for some time τ, if $\| x(\tau) \|_p \le \delta_m$, the linear state feedback control law $u(t) = -Kx(t)$, $t \ge \tau$, satisfies that $u(t) \in \Omega$ for all $t \ge \tau$. ■

The value δ mentioned in Step 2 of the control algorithm is in $(0, \delta_m)$ and δ_m can be obtained by off-line calculation. Since we do not consider systems with a finite escape time, we can assume that there is a compact set $X_{\max} \in R^n$ and $M < \infty$ such that for any $x(t) \in X_{\max}$, there is $u(t)$ satisfying all the constraints of the optimal control problem such that $\|x(t)\| \le M$ for all t. With this assumption, we can show easily that for any $x(t) \in X_{\max}$ there exists $T_{\max} < \infty$ such that $x(t+T) \in B_\delta$ for some $T \le T_{\max}$, which implies that the time interval T_i of a solution of the problem $P(x_i^m)$ is bounded above, i.e. $T_i \le T_{\max}$ for all i.

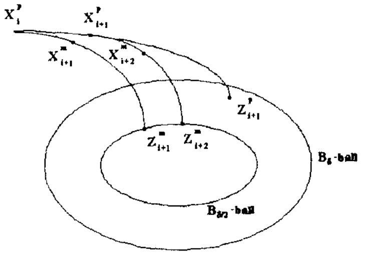

Figure 1. The state trajectories of the real and the model systems

Let us analyze the control algorithm graphically. Figure 1 shows the trajectories of the states. First, at time t_i, the state of the real system x_i^p is measured and the input $u_i(t)$, $t \in [\ t_i,\ t_{i+1}]$ which is the solution of $P(x_i^m)$ is applied to the system. Then, we have two trajectories: one of the real system and the other that is computed using the model. Now, at time t_{i+1}, the state x_{i+1}^m is computed and the optimal control problem $P(x_{i+1}^m)$ is solved to obtain the time interval T_{i+1} and the input $u_{i+1}(t)$, $t \in [\ t_{i+1}, t_{i+1}+T_{i+1}]$. Let us define that $z_{i+1}^m = x^m(t_{i+1}+T_{i+1},\ t_{i+1},\ x_{i+1}^m,\ u_{i+1})$. Then, since due to the modeling error, x_{i+1}^p is different from x_{i+1}^m, at time t_{i+1}, and the model state z_{i+1}^m is different from the state, $z_{i+1}^p = x^m(t_{i+1}+T_{i+1},\ t_{i+1},\ x_{i+1}^p,\ u_{i+1})$. Then, even though z_{i+1}^m is in the set $B_{\delta/2}$ by the constraint of the optimal control problem, z_{i+1}^p may not in the set. We have the following result.

Lemma 3.2: When $x(\tau) \in B_\delta = \{\ z \in R^n \mid \|z\|_p < \delta\}$ at some time τ, suppose that a linear control law $u(t) = -Kx(t)$, $t \ge \tau$, is applied to the system. Then,

(a) for all $t \ge \tau$ and for any $\delta \in (0,\ \delta_m)$, $x(t) \in B_\delta$ is satisfied.

(b) There exists $\Delta_{\max} \in (0,\ \infty)$ such that for any state $x(\tau) \in B_\delta$, $x(\tau+\Delta_{\max}) \in B_{\delta/2}$ is satisfied. ■

We need the following lemma to prove robust stability.

Lemma 3.3: Suppose that Assumption 2.1 and 2.2 are satisifed and at time t_0, $\|x(t)\|$, $\|x^m(t)\| \le M$, $t \ge t_0$, for any intial state $x_0^p \in X_{\max}\ X_0$. Let x_i^p, x_{i+1}^m, x_{i+1}^p, z_{i+1}^m, and z_{i+1}^p be defined as above and let $\beta_{\max} \in (0,\ \infty)$ be the one in Assumption 2.2 (c). Then, for any $\delta \in (0,\ \delta_m)$, there exists $\beta_\delta \in (0,\ \beta_{\max})$ such that for all $\beta \in (0,\ \beta_\delta)$, the following is satisfied.

$$\|z_{i+1}^m - z_{i+1}^p\|_p \le \delta/2 \tag{3}$$

(Proof): Let z_{i+1}^m and z_{i+1}^p be states at time $t_{i+1}+T_{i+1}$ with initial conditions, x_{i+1}^m and x_{i+1}^p, at time t_{i+1}, and an input $u_{i+1}(t)$, $t \in [\ t_{i+1},\ t_{i+1}+T_{i+1}]$ that is obtained by solving the optimal control problem $P(x_{i+1}^m)$, respectively. In other words, if

$$\dot{z}^m(t) = f^m(z^m(t),\ u_{i+1}(t)),\quad z^m(t_{i+1}) = x_{i+1}^m$$

$$\dot{z}^p(t) = f^m(z^p(t),\ u_{i+1}(t)),\quad z^p(t_{i+1}) = x_{i+1}^p$$

it is defined that $z_{i+1}^m = z^m(t_{i+1}+T_{i+1},\ t_{i+1},\ x_{i+1}^m,\ u_{i+1})$, and $z_{i+1}^p = z^p(t_{i+1}+T_{i+1},\ t_{i+1},\ x_{i+1}^p,\ u_{i+1})$. Then, for any $t \in [\ t_{i+1},\ t_{i+1}+T_{i+1}]$ by assumptions 2.1, 2.2, it is satisfied that

$$\begin{aligned}
&\|z^m(t) - z^p(t)\|_p \\
&\le \int_{t_{i+1}}^{t} \|f^m(z^m(\tau),\ u_{i+1}(\tau)) - f^m(z^p(\tau),\ u_{i+1}(\tau))\|_p\, d\tau \\
&\qquad + \|z^m(t_{i+1}) - z^p(t_{i+1})\|_p \\
&\le \int_{t_{i+1}}^{t} L\|z^m(\tau) - z^p(\tau)\|_p\, d\tau \\
&\qquad + \|z^m(t_{i+1}) - z^p(t_{i+1})\|_p
\end{aligned}$$

Then, by Bellman-Gronwall theorem, for any $t \in [\ t_{i+1},\ t_{i+1}+T_{i+1}]$

$$\begin{aligned}
\|z^m(t) - z^p(t)\|_p &\le \|z^m(t_{i+1}) - z^p(t_{i+1})\|_p\, e^{L(t-t_{i+1})} \\
&\le \|x_{i+1}^m - x_{i+1}^p\|_p\, e^{LT_{i+1}}
\end{aligned}$$

is satisfied. And since $T_{i+1} \le T_{\max}$ is asatisfied by the assumption, we have that

$$\|z_{i+1}^m - z_{i+1}^p\|_p \le \|x_{i+1}^m - x_{i+1}^p\|_p\, e^{LT_{\max}}$$

and

$$\begin{aligned}
\|x_{i+1}^m - x_{i+1}^p\| &\le \int_{t_i}^{t_{i+1}} \|f^m(x^m(t),\ u_i(t)) - f(x(t),\ u_i(t))\|_p\, dt \\
&\le \int_{t_i}^{t_{i+1}} \|f^m(x^m(t),\ u_i(t)) - f(x^m(t),\ u_i(t))\|_p \\
&\qquad + \|f(x^m(t),\ u_i(t)) - f(x(t),\ u_i(t))\|_p\, dt
\end{aligned}$$

$$\leq \int_{t_i}^{t_{i+1}} \beta(\|x^m(t)\|_p + \|u_i(t)\|_p) + L\|x^m(t) - x(t)\|_p dt$$

Since $\|x^m(t)\|_p \leq M, \quad t \geq t_0$, by assumption, it is satisfied that

$$\|x_{i+1}^m - x_{i+1}^p\|_p \leq 2\beta M(t_{i+1} - t_i)e^{L(t_{i+1} - t_i)}$$

$$= 2\beta M \Delta e^{L\Delta}$$

Then,

$$\|z_{i+1}^m - z_{i+1}^p\|_p \leq 2\beta M \Delta e^{L(T_{max} + \Delta)}$$

is satisfied. Therefore, if we define

$$\beta_\delta = \min\{\beta_{max}, \ \delta / 4M\Delta e^{L(T_{max}+\Delta)}\}$$

then for any $\beta \in (0, \beta_\delta)$, (3) is satisfied and it completes the proof. ■

Here, β_δ is the maximal norm-difference between the real and the model system, which (3) is satisfied. Now, we are ready to prove the robust stability of the moving horizon control law.

Theorem 3.1: Suppose that (2) is the model system of (1) and Assumption 2.1, 2.2, 3.1, and 3.2 are satisfied. Let the sampling interval Δ satisfiy that. $\Delta > \Delta_{max}$, which is bigger than the time taken to solve the optimal control problem. Then, when a moving horizon control algorithm is applied to the system, it is satisfied that $\|x(t)\|_p \to 0$ as $t \to \infty$. ■

4. Conclusion

In this paper, we showed the robust stability property of the moving horizon control law. The proposed moving horizon control algorithm considers the time taken to solve an optimal control problem so that it can be applied as an on-line control law.

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