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A CLASS OF ASYMPTOTICALLY STABILIZING STATE FEEDBACK CONTROLLERS FOR UNCERTAIN NONLINEAR SYSTEMS

°Yuuki Hashimoto*, Hansheng Wu*, and Koichi Mizukami*

* Faculty of Integrated Arts and Sciences, Hiroshima University
1-7-1 Kagamiyama, Higashi-Hiroshima 739, Japan
Tel: (0824)24-6478; Fax: (0824)24-0756
E-mail: hasimoto@katana.hiroshima-u.ac.jp

Abstract This paper is concerned with the problem of robust stabilization of uncertain single-input and single-output nonlinear systems. Based on the input/output linearization approach for nonlinear state feedback synthesis in conjunction with Lyapunov methods, a stabilizing state feedback controller is proposed. Compared with the controllers reported in the control literature, instead of uniform ultimate boudedness, the controller proposed in this paper can guarantee uniform asymptotic stability of nonlinear systems in the presence of uncertainties. The required information about uncertain dynamics in the system is only that the uncertainties are bounded in Euclidean norm by known functions of the system state.

Keywords Asymptotic Stability, Uncertain Nonlinear Systems, Feedback Linearization, Lyapunov Methods, Matching Condition

1. INTRODUCTION

Many physical systems have uncertain elements; these may be parameter variations, nonlinearities not satisfying closed-form expression of known functions, external disturbances. Thus, it is necessary to design a stabilizing controller such that some types of stability of such systems can be guaranteed in the presence of uncertainties.

Stabilization of a dynamical system with significant uncertainties has widely been studied over the last decade [2, 3, 5, 6]. In [3], for example, for a class of dynamical systems with the so-called matchced uncertainties, a saturation-type controller is designed which guarantees uniform boundedness and uniform ultimate boundedness of such dynamical systems. In [2], a class of linear and nonlinear state feedback controllers are proposed, and the concept of practical stability is introduced. In [5], the input/output linearization approach is applied to synthesize a robust nonlinear state feedback controller that guarantees uniform ultimate boundedness for uncertain nonlinear systems. In [6], a new theory on asymptotic stability is developed, and a class of continuous state feedback controllers are proposed to guarantee asymptotic stability.

In this paper we consider a class of uncertain single-input and single-output nonlinear systems. We propose a class of state feedback controllers for such systems such that instead of uniform ultimate boundedness, uniform asymptotic stability can be guaranteed. The difference between our method and the ones reported in the control literature is that feedback linearization approach is applied to design robust controllers which guarantee asymptotic stability. The novelty of the results obtained in this paper is to be viewed as generalizations of work pioneered by Qu[6].

2. PROBLEM FORMULATION AND SOME ASSUMPTIONS

2.1 Problem Formulation

Consider the SISO nonlinear uncertain systems of the form

$$\begin{aligned} \dot{x} &= (f(x) + \Delta f) + (g(x) + \Delta g)u \\ y &= h(x) + \Delta h \end{aligned} \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R$ is the input, $y \in R$ is the output. $\Delta f, \Delta g$, and Δh represent uncertainties corresponding to $f(\cdot), g(\cdot)$, and $h(\cdot)$, respectively.

In this paper, based on the input/output linearization approach for uncertain nonlinear systems (1), a stabilizing state feedback controller is proposed. The results of input/output linearization is summarized as follows.

2.2 Feedback Linearization

Consider the nominal system of (1) described by the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2}$$

where $x \in R^n$, $u \in R$, $y \in R$ and the state feedback controller can be represented by a nonlinear function

$$u = \Psi(x, v) \tag{3}$$

In the following notations $L_f h(x)$ and $L_g h(x)$ denote the Lie derivative of the function $h(x)$ with respect to the vector field f and that with respect to g, respectively.

$$L_f(h) = \langle dh, f \rangle = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i$$

$$L_g(h) = \langle dh, g \rangle = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} g_i$$

High order Lie derivatives can be defined recursively as follows.

$$L_f^k h(x) = L_f \left(L_f^{k-1} h(x) \right), \ k > 1$$

The Lie bracket $[f, g]$ is a vector field defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

The notation $ad_f g(x)$ denotes the Lie bracket of the vector fields f and g. The following notation is standard.

$$\begin{aligned} ad_f^0(g) &= g \\ ad_f^1(g) &= [f, g] \\ ad_f^2(g) &= [f, [f, g]] \\ &\vdots \\ ad_f^k(g) &= \left[f, ad_f^{k-1}(g) \right] \end{aligned}$$

Consider (2) and the problem of finding a state feedback of the form (3) such that the $v-y$ input/output system is linear and of minimal order. The solution to this problem is summarized in the following lemma.

Lemma 1 [4] Suppose that the minimal order of the $v-y$ system is the relative order of (2), i.e., there exists the smallest integer r satisfying

$$\left\langle dh, ad_f^{r-1}(g) \right\rangle \neq 0$$

The state feedback is of the form

$$u = \frac{v - \sum_{k=0}^{r} \beta_k L_f^k(h)(x)}{(-1)^{r-1} \beta_r \left\langle dh, ad_f^{r-1}(g)(x) \right\rangle} \tag{4}$$

where $\beta_1, \ldots, \beta_r$ are arbitrarily selected numbers. The corresponding input/output closed system is governed by

$$\sum_{k=0}^{r} \beta_k \frac{d^k y}{dt^k} = v$$

2.3 Some Standard Assumptions

We introduce the following standard assumptions.

Assumption 1 f, g, and h are smooth. $\Delta f, \Delta g$, and Δh are C^0 and such that

$$\Delta f \in \Sigma_f$$
$$\Delta g \in \Sigma_g$$
$$\Delta h \in \Sigma_h$$

where Σ_f, Σ_g, and Σ_h are specified sets.

Assumption 2 The origin $x = 0$ is a uniformly asymptotically stable equilibrium point of the unforced nominal closed-loop system

$$\dot{x} = f(x) - \frac{\sum_{k=0}^{r} \beta_k L_f^k(h)(x)}{(-1)^{r-1} \beta_r \left\langle dh, ad_f^{r-1}(g)(x) \right\rangle} g(x)$$

In particular, there exists an C^1 function $V : R^n \to R^+$ and continuous strictly increasing function $\gamma_i : R^+ \to R^+, i = 1, 2, 3$ satisfying

$$\gamma_i(0) = 0, \ i = 1, 2, 3$$
$$\lim_{r \to \infty} \gamma_i(r) = \infty, \ i = 1, 2$$

such that

$$\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$$

$$\left\langle dV, f - \frac{\sum_{k=0}^{r} \beta_k L_f^k(h)}{(-1)^{r-1} \beta_r \left\langle dh, ad_f^{r-1}(g) \right\rangle} \right\rangle \leq -\gamma_3(\|x\|) \tag{5}$$

In other words, there is a Lyapunov function V for the unforced nominal system.

Assumption 3 For each $\Delta f \in \Sigma_f$ and $\Delta g \in \Sigma_g$ there exist mappings $\Delta f^* : R^n \to R$ and $\Delta g^* : R^n \to R$ satisfying

$$\begin{aligned} (-1)^{r-1} \beta_r \left\langle dh, ad_f^{r-1}(g) \right\rangle \Delta f &= g \Delta f^* \\ \Delta g &= g \Delta g^* \end{aligned} \tag{6}$$

for all $x \in R^n$.

Assumption 4 There exists an C^0 function $\mu : R^n \to R$ such that

$$1 + \Delta g^* \geq \mu(x) > 0 \tag{7}$$

for every $\Delta g \in \Sigma_g$. Moreover, $\mu(x)$ is bounded

$$\mu^0 \geq \mu(x) \tag{8}$$

Assumption 5 There exists an C^0 function $\phi : R^n \to R$ such that

$$\phi(x) \geq \frac{1}{\mu(x)} \left| \Delta f^* - \left(\sum_{k=0}^{r} \beta_k L_f^k(h) \right) \Delta g^* \right| \tag{9}$$

for every $\Delta f \in \Sigma_f$ and $\Delta g \in \Sigma_g$.

Assumption 6 There exists $\delta_h > 0$ such that

$$|\Delta h| \leq \delta_h$$

for every $\Delta h \in \Sigma_h$.

In this paper we consider the problem of designing a robust controller which can guarantee uniform asymptotic stability of uncertain nonlinear systems (1) by using feedback linearization (4).

3. ROBUST CONTROLLER

In this section we propose the following feedback controller.

$$u = \frac{v - \sum_{k=0}^{r} \beta_k L_f^k(h)(x)}{(-1)^{r-1}\beta_r \left\langle dh, ad_f^{r-1}(g)(x)\right\rangle} \tag{10}$$

where

$$v = -\phi(x)\eta_\epsilon\left(\frac{\mu^0\phi(x)\left\langle dV, g\right\rangle(x)}{(-1)^{r-1}\beta_r\left\langle dh, ad_f^{r-1}(g)\right\rangle(x)}\right) \tag{11}$$

$$\eta_\epsilon(\xi) = \begin{cases} \dfrac{\xi}{\epsilon\varphi(t)} & \text{if } |\xi| \le \epsilon\varphi(t) \\ \dfrac{\xi}{|\xi|} & \text{if } |\xi| > \epsilon\varphi(t) \end{cases} \tag{12}$$

and $\varphi(t)$ satisfies the following inequalities

$$0 < \varphi(t) \le 1$$
$$\omega(t) \le 0$$

where

$$\omega(t) \triangleq \int \varphi(t)\, dt$$

The following theorem shows the asymptotic stability of system (1) under the control (10).

Theorem Consider system (1) satisfying Assumptions 1 to 6. Let $\lim_{r\to\infty}\inf \gamma_3(r) \triangleq l$. Then, a feedback controller (10) with $\epsilon < 4l$ being a positive definite function of $\|x_0\|$ is continuous and makes every solution of system (1) uniform asymptotically stable in the large.

Proof: Define $\kappa(x)$ and $\lambda(x)$ as

$$\kappa(x) \triangleq \sum_{k=0}^{r}\beta_k L_f^k(h)$$
$$\lambda(x) \triangleq (-1)^{r-1}\beta_r\left\langle dh, ad_f^{r-1}(g)\right\rangle$$

Substituting (10) into (1) we obtain

$$\dot{x} = \left[f - \frac{\kappa}{\lambda}g\right] + \left[\Delta f - \frac{\kappa}{\lambda}\Delta g\right] + \frac{g+\Delta g}{\lambda}v$$

Using (6) we can write

$$\dot{x} = \left[f - \frac{\kappa}{\lambda}g\right] + \frac{g}{\lambda}[\Delta f^* - \kappa\Delta g^*] + \frac{g}{\lambda}[1+\Delta g^*]v$$

Thus

$$\dot{V} = \left\langle dV, f - \frac{\kappa}{\lambda}g\right\rangle + \frac{\langle dV, g\rangle}{\lambda}[\Delta f^* - \kappa\Delta g^*] + \frac{\langle dV, g\rangle}{\lambda}[1+\Delta g^*]v$$

Since $\dot{V}$ is a scalor function, the following equation is satisfied.

$$\dot{V} \le -\gamma_3(\|x\|) + \left|\frac{\langle dV, g\rangle}{\lambda}\right||\Delta f^* - \kappa\Delta g^*| + \frac{\langle dV, g\rangle}{\lambda}[1+\Delta g^*]v$$

Substituting v from (11)

$$\dot{V} \le -\gamma_3(\|x\|) + \left|\frac{\langle dV, g\rangle}{\lambda}\right||\Delta f^* - \kappa\Delta g^*| - \left|\frac{\phi\langle dV, g\rangle}{\lambda}\right|^2[1+\Delta g^*]\frac{\mu^0}{\Theta_\epsilon\left(\frac{\mu^0\phi\langle dV,g\rangle}{\lambda}\right)}$$

where

$$\Theta_\epsilon(\xi) = \begin{cases} \epsilon\varphi(t) & \text{if } |\xi| \le \epsilon\varphi(t) \\ |\xi| & \text{if } |\xi| > \epsilon\varphi(t) \end{cases}$$

Using (7), (8) and (9)

$$\dot{V} \le -\gamma_3(\|x\|) + \frac{\mu}{\mu^0}\left|\frac{\mu^0\phi\langle dV, g\rangle}{\lambda}\right| - \frac{\mu}{\mu^0}\left|\frac{\mu^0\phi\langle dV, g\rangle}{\lambda}\right|^2\frac{1}{\Theta_\epsilon\left(\frac{\mu^0\phi\langle dV,g\rangle}{\lambda}\right)}$$

Thus, if $|\xi| > \epsilon\varphi(t)$

$$\dot{V} \le -\gamma_3(\|x\|)$$

and if $|\xi| \le \epsilon\varphi(t)$

$$\dot{V} \le -\gamma_3(\|x\|) + \frac{1}{4}\epsilon\varphi(t)$$

by using the following equation

$$|\xi| - \frac{|\xi|^2}{\epsilon\varphi(t)} \le \frac{1}{4}\epsilon\varphi(t)$$

Consequently, we conclude

$$\dot{V} \le -\gamma_3(\|x\|) + \frac{1}{4}\epsilon\varphi(t)$$

Here, we introduce the following lemma to complete the remainder of the proof.

Lemma 2 [6] Suppose that $V(\cdot) : R^n \times R \to R^+$ is a Lyapunov function candidate for any given continuous time system with the following properties:

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|)$$
$$\dot{V}(x,t) \le -\gamma(\|x\|) + \gamma(\eta)\varphi(t)$$

where $\eta > 0$ is a constant, $\lim_{p\to\infty}\gamma_j(p) = \infty, j = 1, 2, \gamma_1$ and γ_2 are those defined in *Assumption 2*, γ is continuous and positive definte, and $\gamma(0) = 0$. Let $\omega(t) \triangleq \int\varphi(t)\,dt$ denote the indefinte integral function of $\varphi(t)$. If function γ, $\varphi(t)$, and $\omega(t)$ satisfy $\gamma(y) - \gamma(\eta) > 0$ whenever $y > \eta$, $0 < \varphi(t) \le 1$, $\omega(t) \le 0$, and if the parameter η can be chosen to be a positive definte function of $\|x_0\|$, then, every continuous solution $x(t; x_0, t_0) : [t_0, \infty] \to R^n$ of the given system, with initial state $x(t_0) = x_0$, is asymptotically stable in the large.

By using above lemma, we can obtain the results in this theorem. ■

4. ILLUSTRATIVE EXAMPLE

Consider the following system to demonstarate the validity our results.

$$\begin{aligned}\dot{x} &= (f(x)+\Delta f)+(g(x)+\Delta g)u\\ y &= h(x)+\Delta h\end{aligned}$$

where

$$f(x) = \begin{bmatrix}\dfrac{-10x_1 - x_1x_2 + x_2}{10+x_2}\\ \dfrac{x_1x_2}{10}\end{bmatrix}$$

$$g(x) = \begin{bmatrix}0\\1\end{bmatrix} \quad h(x) = \frac{x_2}{10+x_2}$$

$$\Delta f = \begin{bmatrix}0\\-0.05x_2\end{bmatrix}$$

$$\Delta g = \begin{bmatrix}0\\0.3\end{bmatrix}$$

It can be easily verified that $< dh, g > \neq 0$ and thus the relative order of the system is 1. Thus, from (10), we let

$$u = \frac{v - \beta_0\left(\frac{x_2}{10+x_2}\right) - \beta_1\left(\frac{x_1x_2}{10(10+x_2)^2}\right)}{\frac{\beta_1}{(10+x_2)^2}}$$

where $\beta_0 = 1.0, \beta_1 = 1.0$. Consider the following positive definte function as a Lyapunov function candidate for unforced nominal system.

$$V = 2x_1^2 + \frac{2x_1x_2}{10+x_2} + 3\left(\frac{x_2}{10+x_2}\right)^2$$

Then V has the time derivative

$$\dot{V} = -4x_1^2 - 4\left(\frac{x_2}{10+x_2}\right)^2$$

which is negative definite. Thus, V is a desirable Lyapunov function.

Since

$$\begin{aligned}\Delta f^* &= \frac{-0.05x_2}{(10+x_2)^2}\\ \Delta g^* &= 0.3\end{aligned}$$

from (7) and (8), let $\mu(x) = \mu^0 = 0.3$.

From (9), let

$$\phi(x) = \frac{1}{0.3}\left|\frac{-0.05x_2}{(10+x_2)^2} - \frac{0.3x_2}{(10+x_2)} - \frac{0.03x_1x_2}{(10+x_2)^2}\right|$$

Then

$$v = -\phi(x)\eta_\epsilon\left(0.3\left[10 + \frac{6x_2}{(10+x_2)}\right]\phi(x)\right)$$

where

$$\eta_\epsilon(\xi) = \begin{cases}\dfrac{\xi}{\epsilon\exp(-\alpha t)} & \text{if } |\xi| \leq \epsilon\exp(-\alpha t)\\ \dfrac{\xi}{|\xi|} & \text{if } |\xi| > \epsilon\exp(-\alpha t)\end{cases}$$

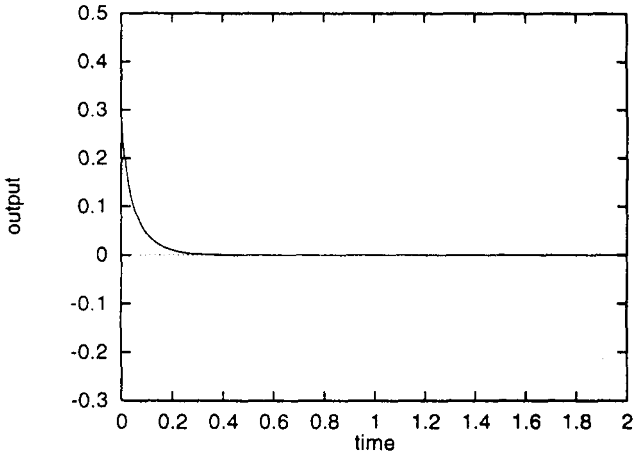

Fig. 1. The response of a output in Example with initial condition $x(0) = [4\ 4]^t$.

Pick $\epsilon = 1.00, \alpha = 0.01$. Then, the results of simulation are shown in Fig. 1.

It is observed in Fig. 1 that the output of the system is asymptotically stable, as predicted.

5. CONCLUSION

The problem to design a stabilizing controller for uncertain nonlinear systems which guarantees asymptotic stability has been considered. For such a problem, we have proposed a calss of robust controllers by feedback linearization and Lyapunov methods. The difference between our method and the ones reported in the control literature is that feedback linearization approach is applied to design robust controllers which guarantee asymptotic stability instead of uniform ultimate boundedness.

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