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ROBUST STABILITY ANALTSIS FOR A FLEXIBLE ARM

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Abstracts We investigate the applicability of the theory of robust stabilization with respect to additive, stable perturbations of a normalized left-coprime factorization to controller design of a flexible arm with uncertain parameters.

Key words Robust stability, Flexible arm, T-gap

1. INTRODUCTION

The problem of robust stabilization with respect to additive stable perturbation of a normalized left-coprime factorization has been considered by several researchers[1]-[4]. In [1] it was shown that for the rational case this problem has an elegant explicit solution in term of Riccati equations and in [4] this problem is extended to a class of infinite-dimensional systems. In [3] it was shown that the problem of robustness optimization for normalized coprime factor perturbation is equivalent to robust optimization in the gap metric.

However, these theories only consider unstructured perturbations, whereas in flexible structures one usually has to take structured perturbations into account. We consider a prototype example of a p.d.e. model of a damped arm in which we suppose that the damping coefficient are unknown. This p.d.e. model retains some essential characteristic, typical of large flexible structures, such as uncertain damping and point actuators and sensors, while at the same time it is possible to obtain a rigorous mathematical formulation in both time and frequency domain. In particular, it belongs to the class of infinite-dimensional systems discussed in [4].

In absence of a theory for robustness optimization under structured perturbations for infinite-dimensional systems, we decided to investigate how unstructured theories of robustness optimization would work on this prototype model. We apply the theory of robustness optimization with respect to additive stable perturbations of a normalized left-coprime factorization.

The first part of investigation is the dependence of the maximal robustness margin on the damping parameters. Then taking a fixed parameter value to define the nominal model, we designed the controller which achieves the (nearly) maximal robustness margin. Then we evaluate the range of parameter value in which this controller maintains closed-loop stability. The range of parameter value can be calculated in term of the T-gap between the nominal plant and the perturbed plant.

For the infinite-dimensional theory of robustness optimization with respect to additive stable perturbations of a normalized left-coprime factorization we refer to the paper [4]. The brief outline of [1] is shown in the section 2. In the section 3 we summarized the relevant results on relationship to the T-gap metric from [3]. In the section 4 we explain a flexible arm which is considered in this paper. Several numerical results is shown in the section 5. Finally some conclusion remarks is give in the section 6.

We use following notations in this paper. A state space realization of transfer function G is denoted by

$$G = (A\,,B\,,C\,,D)$$

or

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $G(s) = D + C(sI - A)^{-1}B$. The space consisting of all real-rational, proper, transfer function is denoted by RH_∞ and $\|\cdot\|_\infty$ is denotes the H_∞-norm. Let P be the controllability gramian of $G \in RH_\infty$ and Q be the observability gramian of G. Then Hankel norm of G, denoted by $\|G\|_H$, is defined as

$$\|G\|_H = \lambda_{max}(PQ)$$

where $\lambda_{max}(\cdot)$ denotes the largest eigenvalue.

2. ROBUST STABILIZATION FOR NORMALIZED COPRIME FACTOR

This is a brief outline of the robust stabilization problem for a normalized coprime factor plant description[1].

Given a nominal plant model with transfer function $G(s)=\tilde{M}^{-1}(s)\tilde{N}(s)$ where $\tilde{M},\tilde{N}\in H_\infty$, a suitable class of perturbed plant is defined

$$\mathcal{G}_\varepsilon=\left\{G_\Delta=(\tilde{M}+\Delta_M)^{-1}(\tilde{N}+\Delta_N):\Delta_M,\Delta_N\in RH_\infty,\left\|\left[\Delta_M\ \Delta_N\right]\right\|_\infty<\varepsilon\right\} \quad (1).$$

In the case when coprime factors are normalized, that is

$$\tilde{M}\tilde{M}^*+\tilde{N}\tilde{N}^*=1$$

the robust stabilization problem for a perturbed plant $G_\Delta\in\mathcal{G}_\varepsilon$ has following explicit solution.

There exists a controller K which stabilizes all plants $G_\Delta\in\mathcal{G}_\varepsilon$ if and only if

$$\varepsilon\le\left(1-\left\|\left[\tilde{N}\ \tilde{M}\right]\right\|_H^2\right)^{1/2} \quad (2)$$

where $\|\cdot\|_H$ denotes the Hankel norm. (2) means that he maximal robustness margin ε_{max} is given by

$$\varepsilon_{max}=\left(1-\left\|\left[\tilde{N}\ \tilde{M}\right]\right\|_H^2\right)^{1/2} \quad (3).$$

Further more, for $\varepsilon\le\varepsilon_{max}=\left(1-\left\|\left[\tilde{N}\ \tilde{M}\right]\right\|_H^2\right)^{1/2}$ a state-space realization of a controller K which stabilizes all plant $G_\Delta\in\mathcal{G}_\varepsilon$ is obtained as follows.

$$K=\left[\begin{array}{c|c}A^C+\gamma^2U^{*-1}ZC^*(C+DF) & U^{*-1}ZC^* \\ \hline B^*X & -D^*\end{array}\right] \quad (4)$$

where $\gamma=\varepsilon^{-1}$, $G=\left[\begin{array}{c|c}A&B\\ \hline C&D\end{array}\right]$, $U=I+(XZ-\gamma^2I)$, $S=I+D^*D$, $F=-S^{-1}(B^*X+D^*C)$ and $A^C=A+BF$. In (4) X and Z are solution to $\tilde{A}^*X+X\tilde{A}-XBS^{-1}B^*X+C^*R^{-1}C=0$ and $\tilde{A}Z+Z\tilde{A}^*-ZC^*R^{-1}CZ+BS^{-1}B^*=0$ respectively where $R=I+DD^*$, $S=I+D^*D$ and $\tilde{A}=A-BS^{-1}D^*C$.

3. OPTIMAL ROBUSTNESS AND THE GAP METRIC

Here we summarized results from [3] which we need in sequel. For simplicity we suppose that P_1 and P_2 are in the Prichard-Salamon class defined in [4]. The directed T-gap is defined by

$$\vec{\delta}_T(P_1,P_2)=\inf_{Q\in H_\infty}\left\|\left[\tilde{M}_1,\tilde{N}_1\right]-Q\left[\tilde{M}_2,\tilde{N}_2\right]\right\|_\infty \quad (5)$$

and T-gap is defined by

$$\delta_T=\max\left\{\vec{\delta}_T(P_1,P_2),\vec{\delta}_T(P_2,P_1)\right\} \quad (6)$$

where $P_i=\tilde{M}_i^{-1}\tilde{N}_i$ is a normalized left-coprime factorization of P_i $(i=1,2)$.

In general $\vec{\delta}_T(P_1,P_2)\ne\vec{\delta}_T(P_2,P_1)$, but if $\vec{\delta}_T(P_1,P_2)<1$, then they are both equal. There exists the following relationship between the Glover-McFarlane class of perturbations $\mathcal{G}_\varepsilon$ in (1) and the directed T-gap ball:

$$B_T(P,\varepsilon)=\left\{P_1:\vec{\delta}_T(P,P_1)<\varepsilon\right\}=\left\{P_1=(\tilde{M}^{-1}+\Delta_M)^{-1}(\tilde{N}+\Delta_N)\right.$$

$$:P=\tilde{M}^{-1}\tilde{N}\text{ is a normalized coprime factorization,} \quad (7)$$

$$\left.\Delta_M,\Delta_N\in H_\infty,\left\|\left[\Delta_M\ \Delta_N\right]\right\|_\infty<\varepsilon\right\}$$

The main result in [3] is that a controller K stabilizes all P_1 with $\vec{\delta}_T(P,P_1)<\varepsilon$, if and only if K stabilizes all $P_1=(\tilde{M}+\Delta_M)^{-1}(\tilde{N}+\Delta_N)$ where $\Delta_M,\Delta_N\in H_\infty$ satisfy $\left\|\left[\Delta_M\ \Delta_N\right]\right\|_\infty<\varepsilon$. The advantage of the gap metric is that it can be calculate (at least for finite-dimensional plan) as a 2-block H_∞-optimization problem[3],[5].

4. Flexible Arm model and approximation

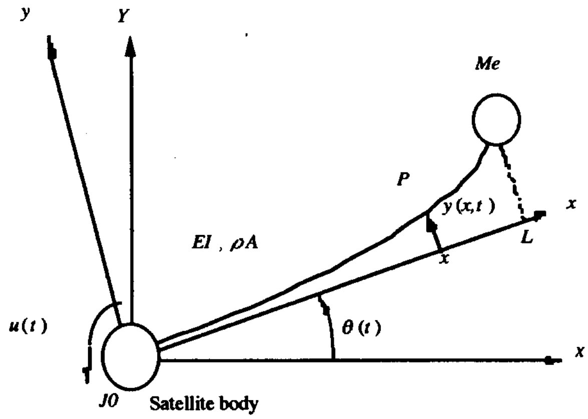

Fig.1 Schematic Diagram of the Arm

Table 1 Notation in Fig. 1

J_0	Moment of inatia of the satellite body
L	Length of the arm
ρA	Linear Densiry of the Arm
EI	Flexurel Rigidity
M_e	Lumped Mass at the Tip of the Arm
J_e	Moment of inatia of the Lumped Mass at the Tip of the Arm
$u(t)$	Applied Torque
$\theta(t)$	Angle of the Arm

In this paper we consider a flexible arm shown in Fig. 1. A mathematical model of the arm shown in Fig. 1 is given by following equations.

$$J_r\theta(t)+\rho A\int_0^L x(x,y)\,dx+M_eLy(L,t)=u(t) \quad (8)$$

$$EI\frac{\partial^4y(x,t)}{\partial t^4}+\rho A\left\{L\theta(t)+y(x,t)\right\}=0 \quad (9)$$

The boundary conditions are given by

$$EI\frac{\partial^3 y(x,t)}{\partial t^3} = M_e\left\{L\theta(t) + y(L,t)\right\} \quad (10)$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (11).$$

Using the eigenfunction corresponding to the boundary conditions the transfer function from applied torque to the angle of the arm is obtained as follow.

$$G(s) = \frac{A_0}{s} + \sum_{n=1}^{\infty} \frac{A_i}{s^2 + \alpha\omega_i s + \omega_i^2} \quad (12)$$

Because it has been shown in [4] that this type of flexible arm model has a Prichard-Salamon state-space realization, we may apply the results of [4] directly. However, this would involve solving infinite-dimensional Riccati equations, which is very time consuming at best and in fact no known convegence results for the Riccati equations of our example. They have so-called "unbounded" B and C operators and very little work has been done on the numerical approximation of solutions of such Riccati equations.

So we shall take an approximation approach using the known properties of the transfer function (12). From (12) G represents the sum of an finite-dimensional part G_f which contains the unstable mode and infinite-dimensional stable part G_s. Further G_s is nuclear, which means that it is easy to approximate by a finite-dimensional system and bound on error can be calculated[4]. For our investigation we obtain the following G_f.

$$G_f(s) = \frac{A_0}{s} + \sum_{n=1}^{2} \frac{A_i}{s^2 + \alpha\omega_i s + \omega_i^2} \quad (13)$$

For the nominal value of $\alpha = 4.769 \times 10^{-2}$. We have a bound on error μ as follow.

$$\mu = \left\| G - G_f \right\|_\infty = \left\| G_s \right\|_\infty \leq 0.02 \quad (14)$$

G_f is rational and we can apply the results of [4] to G_f allowing for infinite-dimensional perturbations. If G_f has a normalized coprime factorization

$$G_f = \tilde{M}_f^{-1}\tilde{N}_f \quad (15)$$

then

$$G = G_f + G_s = \tilde{M}_f^{-1}\left(\tilde{N}_f + \Delta_N^f\right) \quad (16)$$

where

$$\left\|\Delta_N^f\right\|_\infty \leq \left\|\tilde{M}_f G_s\right\|_\infty \leq \left\|G_s\right\|_\infty < \mu \quad (17)$$

From (17), we may conclude that

$$\vec{\delta}_T(G_f, G) < \mu < 1 \quad (18)$$

Reversing the role of G and G_f, we can write

$$G_f = G - G_s = \tilde{M}^{-1}(\tilde{N} + \Delta_N) \quad (19)$$

where

$$\left\|\Delta_N\right\|_\infty \leq \left\|\tilde{M} G_s\right\|_\infty \leq \left\|G_s\right\|_\infty < \mu \quad (20)$$

Hence by applying the same argument above we can conclude

$$\vec{\delta}_T(G, G_f) < \mu < 1 \quad (21)$$

Therefore

$$\vec{\delta}_T(G_f, G) = \vec{\delta}_T(G, G_f) \quad (22)$$

Suppose that we apply the finite-dimensional theory of [1] on G_f to obtain a controller K_f with robustness margin ε. Then from the results quoted in the section 2, we see that K_f is a robust controller for G with a robustness margin of at least $\varepsilon - \mu$. In other word, replacing G by G_f in our calculations incurs an error of at most μ and we have chosen to be negligible compared to the robustness margin of G_f for our range of parameter values. This justifies using G_f in our calculations.

5. Numerical Results

We first have considered the dependence of the maximal robustness margin ε_{max} both on the order of approximation of the infinite-dimensional system and on the parameter α.

Table 2 shows the dependence on the order of approximation of the infinite-dimensional system. ($\alpha = 4.769 \times 10^{-2}$ which is equal to the nominal values.) In Table 2 n denotes the highest flexible mode which is included in the nominal plant and $n = 0$ means that the nominal plant only includes the rigid body mode dynamics.

The dependence on the parameter α is shown in Table 3. (The nominal plant includes the rigid body dynamics and the first two flexible modes dynamics.)

As shown in Table 2 and Table 3, the maximal robustness margin ε_{max} does not depend on neither the order of approximation of the infinite-dimensional system nor on the parameter α drastically.

For the nominal plant which includes the rigid body dynamics and the first two flexible modes dynamics, we choose as the nominal value $\alpha = 4.769 \times 10^{-2}$. Then parameter α is varied and we calculate the distance between the perturbed plan and the nominal plant. If the distance (the directed T-gap) between the nominal plant and perturbed plant is smaller than the maximal robustness margin ε_{max}, then the both plant are guaranteed to be stabilized by a maximally robust controller which is designed for the nominal plant[1]. In order to calculate the directed T-gap we have to solve a 2-block H_∞-control problem which can be a numerically hard problem.

For this reason we used the following results for the directed T-gap[5]: Let G_1 and G_2 have the normalized left and right coprime factorization.

$$G_1 = \tilde{M}_1^{-1}\tilde{N}_1 = N_1 M_1^{-1},\ G_2 = \tilde{M}_2^{-1}\tilde{N}_2 = N_2 M_2^{-1}$$

Define

$$\tilde{R}_1 = \tilde{M}_1\tilde{M}_2^* + \tilde{N}_1\tilde{N}_2^*,\ R_2 = \tilde{M}_1 N_2 - \tilde{N}_1 M_2$$

then

$$\max\left(\left\| R_2 \right\|_\infty, \tilde{H}_{R1}\right) \le \vec{\delta}_T(G_1, G_2) \le \left(\left\| R_2 \right\|_\infty + \tilde{H}_{R1}^2\right)^{1/2} \qquad (23)$$

where $\tilde{H}_{R1} = \inf_{Q \in H_\infty} \left\| \tilde{R}_1 - Q \right\|_\infty$.

For different values of α, both the upper and lower bounds for T-gap between the nominal plant and perturbed plant are shown in Table 4.

Table 2 Dependence ε_{max} on the approximation order

n	0	1	2	3
ε_{max}	0.2256	0.3916	0.3924	0.3931

Table 3 Dependence ε_{max} on parameter α

α	4.769×10^{-4}	0.5299×10^{-3}	0.5961×10^{-3}
ε_{max}	0.3922	0.3922	0.3922
α	0.6813×10^{-3}	0.7948×10^{-3}	0.9538×10^{-3}
ε_{max}	0.3922	0.3923	0.3923
α	1.1923×10^{-3}	1.5897×10^{-3}	2.3845×10^{-3}
ε_{max}	0.3923	0.3924	0.3924
α	4.869×10^{-3}	4.869×10^{-2}	
ε_{max}	0.3924	0.3924	

Table 4 The upper and lower bounds for T-gap between the nominal plant and perturbed plant

α	4.769×10^{-4}	0.5299×10^{-3}	0.5961×10^{-3}
T-gap	0.9618	0.8329	0.6791
	0.9542	0.8222	0.6633
α	0.6813×10^{-3}	0.7948×10^{-3}	0.9538×10^{-3}
T-gap	0.4351	0.2154	0.2133
	0.4217	0.2153	0.2132
α	1.1923×10^{-3}	1.5897×10^{-3}	2.3845×10^{-3}
T-gap	0.2154	0.2092	0.1973
	0.2153	0.2091	0.1973
α	4.869×10^{-3}	4.869×10^{-2}	
T-gap	0.1899		
	0.1899		

For the nominal plant which includes the rigid body dynamics and the first two flexible modes dynamics, the maximal robustness margin ε_{max} is equal to 0.3924. ($n = 2$ in Table 2) Comparing this ε_{max} (= 0.3924) with the T-gap as shown in Table 4, we can find the rang of parameter values in which a maximally robust controller, designed for the nominal plant, is guaranteed to stabilize the perturbed plant. From Table 4 it can be seen that the maximally robust controller allows parameter α to change from $\alpha = 4.769 \times 10^{-2}$ (the nominal value) to $\alpha = 0.7948 \times 10^{-3}$.

6. CONCLUSIONS

In this paper we consider the robust stability analysis for a flexible arm using T-gap. In particular we investigate the applicability of robustness optimization theories for the unstructured perturbation to the structured perturbation. The theory of robust stabilization with respect to additive, stable perturbations of a normalized corime factorization proposed in [1] seems to be a useful method for designing controller even when the perturbations are structured.

The first our investigation is the dependence of the maximal robustness margin on the damping parameters. Then taking a fixed parameter value to define the nominal model, we designed the controller which achieves the (nearly) maximal robustness margin. Then we evaluate the range of parameter value in which this controller maintains closed-loop stability. From the numerical results the maximal robustness margin does not depend on neither the order of approximation of the infinite-dimensional system nor on the parameter α drastically and the maximally robust controller which is designed for the nominal plant allows the damping parameter α to change from $\alpha = 4.769 \times 10^{-2}$ (the nominal value) to $\alpha = 0.7948 \times 10^{-2}$.

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