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THE RECURSIVE ALGORITHM FOR OPTIMAL REGULATOR OF NONSTANDARD SINGULARLY PERTURBED SYSTEMS

Hiroaki Mukaidani* , Hua Xu** and Koichi Mizukami**

*Graduate School of Engineering, Hiroshima University,
1-7-1, Kagamiyama, Higashi-Hiroshima, Japan 739

**Faculty of Integrated Arts and Sciences, Hiroshima University,
1-7-1, Kagamiyama, Higashi-Hiroshima, Japan 739

Abstract: This paper considers the linear-quadratic optimal regulator problem for nonstandard singularly perturbed systems making use of the recursive technique. We first derive a generalized Riccati differential equation by the Hamilton-Jacobi equation. In order to obtain the feedback gain , we must solve the generalized algebraic Riccati equation. Using the recursive technique, we show that the solution of the generalized algebraic Riccati equation converges with the rate of convergence of $O(\varepsilon)$. The existence of a bounded solution of error term can be proved by the implicit function theorem. It is enough to show that the corresponding Jacobian matrix is nonsingular at $\varepsilon = 0$.

As a result, the solution of optimal regulator problem for nonstandard singularly perturbed systems can be obtained with an accuracy of $O(\varepsilon^k)$. The proposed technique represents a significant improvement since the existing method for the standard singularly perturbed systems can not be applied to the nonstandard singularly perturbed systems.

KeyWords: Nonstandard singularly perturbed systems, Generalized algebraic Riccati equation, Recursive algorithm, Implicit function theorem.

1. INTRODUCTION

We consider a singularly perturbed linear time-invarant system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \tag{1a}$$

$$\varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u \tag{1b}$$

where ε is a small positive parameter, $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are states, $u(t) \in R^m$ is the control. The system (1a)~(1b) is called the nonstandard singularly perturbed systems if the matrix A_{22} is singular.

We find the optimal control $u(t), t \in [0, \infty]$,which minimizes

$$J = \min_u\{\frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt\}, \tag{2}$$

In this paper we study the linear-quadratic optimal regulator problem for nonstandard singularly perturbed systems by making use of the recursive technique. We first derive a generalized Riccati differntial equation by the Hamilton-Jacobi equation.

2. GENERALIZED RICCATI ALGEBRAIC EQUATION

At first, we define

$$D = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix} \tag{3a}$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3b}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{3c}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{3d}$$

We consider the linear-quadratic optimal control problem for the nonstandard singularly perturbed systems, that is

$$D\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

Now assume that the optimal perfomance index for the problem takes the form $V^*(Dx(t), t) = (1/2)x^T D^T P(t)x$, when the initial variable is $x(t)$ at time t, where the $(n_1 + n_2) \times (n_1 + n_2)$ time-varying matrix $P(t)$ satisfies the condition $D^T P(t) = P^T(t)D$.

Define

$$L(x(t), u(t), t) = (1/2)(x^T Qx + u^T Ru) \tag{5a}$$

$$f(x(t), u(t), t) = Ax + Bu \tag{5b}$$

$$W^*(x(t), t) = x^T P^T(t) \tag{5c}$$

sinse $D^T P = P^T D$.

By making use of Hamilton-Jacobi equation [7], i.e.

$$\frac{\partial V^*}{\partial t} = -\min_{u(t)}\{L(x(t), u(t), t) + W^* f(x(t), u(t), t)\} \tag{6}$$

we have

$$x^T D^T \dot{P} x = -\min_{u(t)}\{x^T Qx + u^T Ru + 2x^T P^T(Ax + Bu)\} \tag{7}$$

where $(\partial V^*/\partial t) = (1/2)x^T D^T \dot{P} x$ sinse $D^T P = P^T D$.

Carring out minimization on the right-hand side of (7) gives

$$u^*(t) = -R^{-1}B^T P(t)x(t) \tag{8}$$

Substituting (8) into (7) yield

$$x^T D^T \dot{P} x = -x^T[Q + A^T P + P^T A - P^T BR^{-1}B^T P]x \tag{9}$$

The equation holds for all $x(t)$. Therefore, we obtain a generalized Riccati differential equation

$$D^T \dot{P} = -Q - A^T P - P^T A + P^T BR^{-1}B^T P \tag{10a}$$

$$D^T P = P^T D \tag{10b}$$

Since the infinite-horizen problem can be considerd as a limiting case of the finite-horizen problem, we can obtain a generalized Riccati algebraic equation by letting $\dot{P}(t) = 0$.

$$A^T P + P^T A - P^T BR^{-1}B^T P + Q = 0 \tag{11a}$$

$$D^T P = P^T D \tag{11b}$$

3. RECURSIVE ALGOLITHM

Consider the generalized Riccati algebraic equation (11a)~(11b). Partitioning (11b) subject to (3a) we get the followig equations

$$\begin{aligned}
& D^T P = P^T D \\
\Leftrightarrow\ & \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11}^T & P_{21}^T \\ P_{12}^T & P_{22}^T \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix} \\
\Leftrightarrow\ & P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^T \\ P_{21} & P_{22} \end{bmatrix} \\
& P_{11} = P_{11}^T, P_{22} = P_{22}^T.
\end{aligned} \tag{12}$$

Partitioning (11a) subject to (3b)~(3d) we also get the followig equations

$$A_{11}^T P_{11} + P_{11}^T A_{11} + A_{21}^T P_{21} + P_{21}^T A_{21} - P_{11}^T S_{11} P_{11} - P_{21}^T S_{22} P_{21} - P_{11}^T S_{12} P_{21} - P_{21}^T S_{12}^T P_{11} + Q_{11} = 0 \tag{13a}$$

$$\varepsilon P_{21} A_{11} + P_{22}^T A_{21} + A_{12}^T P_{11} + A_{22}^T P_{21} - \varepsilon P_{21} S_{11} P_{11} - \varepsilon P_{21} S_{12}^T P_{21} - P_{22}^T S_{12}^T P_{11} - P_{22}^T S_{22} P_{21} + Q_{12}^T = 0 \tag{13b}$$

$$A_{22}^T P_{22} + P_{22}^T A_{22} + \varepsilon A_{12}^T P_{21}^T + \varepsilon P_{21} A_{12} - P_{22}^T S_{22} P_{22} - \varepsilon P_{22}^T S_{12}^T P_{21}^T - \varepsilon P_{21} S_{12} P_{22} - \varepsilon^2 P_{21} S_{11}^T P_{21}^T + Q_{22} = 0 \tag{13c}$$

where

$$\begin{aligned}
S_{11} &= B_1 R^{-1} B_1^T \\
S_{12} &= B_1 R^{-1} B_2^T \\
S_{22} &= B_2 R^{-1} B_2^T
\end{aligned}$$

Setting $\varepsilon = 0$, we obtain the following equations

$$A_{11}^T P_{11} + P_{11}^T A_{11} + A_{21}^T P_{21} + P_{21}^T A_{21} - P_{11}^T S_{11} P_{11} - P_{21}^T S_{22} P_{21} - P_{11}^T S_{12} P_{21} - P_{21}^T S_{12}^T P_{11} + Q_{11} = 0 \tag{14a}$$

$$P_{22}^T A_{21} + A_{12}^T P_{11} + A_{22}^T P_{21} - P_{22}^T S_{12}^T P_{11} - P_{22}^T S_{22} P_{21} + Q_{12}^T = 0 \tag{14b}$$

$$A_{22}^T P_{22} + P_{22}^T A_{22} - A_{22}^T S_{22} P_{22} + Q_{22} = 0 \tag{14c}$$

The Riccati equation (14c) will produce the unique positive semidefinite stabilizing solution under the following assumption.

Assumption 1. The triple(A_{22}, B_2, Q_{22}) is stabilizable and detectable.

The matix $A_{22} - S_{22}P_{22}$ is nonsingular if Assumption 1 hold. Therefore, we obtain the following 0-order equations

$$\bar{P}_{11}^T A_0 + A_0^T \bar{P}_{11} - \bar{P}_{11}^T S_0 \bar{P}_{11} + Q_0 = 0 \tag{15a}$$

$$\bar{P}_{21} = -N_2^T + N_1^T \bar{P}_{11} \tag{15b}$$

$$A_{22}^T \bar{P}_{22} + \bar{P}_{22}^T A_{22} - \bar{P}_{22}^T S_{22} \bar{P}_{22} + Q_{22} = 0 \tag{15c}$$

where

$$\begin{aligned} A_0 &= A_{11} + N_1 A_{21} + S_{12} N_2^T + N_1 S_{22} N_2^T \\ S_0 &= S_{11} + N_1 S_{12}^T + S_{12} N_1^T + N_1 S_{22} N_1^T \\ Q_0 &= Q_{11} - N_2 A_{21} - A_{21}^T N_2^T - N_2 S_{22} N_2^T \\ N_2^T &= D_4^{-T} \hat{Q}_{12}^T,\ N_1^T = -D_4^{-T} D_2^T \\ D_2 &= A_{12} - S_{12}\bar{P}_{22},\ D_4 = A_{22} - S_{22}\bar{P}_{22} \\ \hat{Q}_{12} &= Q_{12} + A_{21}^T \bar{P}_{22} \end{aligned}$$

The unique positive semidefinite stabilizing solution of (15a) exists under the following assumption.

Assumption 2. The triple(A_0, S_0, Q_0) is stabilizable and detectable.

Note. Although the expressions of the matrix A_0, S_0 and Q_0 contain the matrix $\bar{P}_{22}$, they do not depend on it.

The 0-order solution is $O(\varepsilon)$ close to the exact one. We define errors as

$$P_{11} = \bar{P}_{11} + \varepsilon E_{11} \quad (16a)$$

$$P_{21} = \bar{P}_{21} + \varepsilon E_{21} \quad (16b)$$

$$P_{22} = \bar{P}_{22} + \varepsilon E_{22} \quad (16c)$$

The $O(\varepsilon^k)$aproximation of E will produced the $O(\varepsilon^{k+1})$ approximation of the required matrix P, which is why we are interested in finding equations for the error term and a convenient algorithm for its solution. Subtracting (15a)~(15c) from (13a)~(13c) and using (16a)~(16c) we arrive at the following expression for the error equation.

$$\begin{aligned} &E_{11}^T D_0 + D_0^T E_{11} + V^T H_1^T + H_1 V \\ &\quad - V^T H_3 V - \varepsilon H_2 = 0 \end{aligned} \quad (17a)$$

$$E_{11}^T D_2 + E_{21}^T D_4 + D_3^T E_{22} - H_1 = 0 \quad (17b)$$

$$E_{22}^T D_4 + D_4^T E_{22} - H_3 = 0 \quad (17c)$$

where

$$\begin{aligned} H_1 &= -A_{11}^T P_{21}^T + P_{11}^T S_{11} P_{21}^T + P_{21}^T S_{12}^T P_{21}^T \\ &\quad +\varepsilon(E_{11}^T S_{12} E_{22} + E_{21}^T S_{22} E_{22}) \\ H_2 &= E_{11}^T S_{11} E_{11} + E_{21}^T S_{22} E_{21} \\ &\quad + E_{11}^T S_{12} E_{21} + E_{21}^T S_{12}^T E_{11} \\ H_3 &= -A_{12}^T P_{21}^T - P_{21} A_{12} + \varepsilon P_{21} S_{11} P_{21}^T \\ &\quad +\varepsilon E_{22}^T S_{22} E_{22} + P_{21} S_{12} P_{22} \\ &\quad + P_{22}^T S_{12}^T P_{21}^T \end{aligned}$$

and

$$\begin{aligned} D_0 &= D_1 - D_2 D_4^{-1} D_3,\ V = D_4^{-1} D_3 \\ D_1 &= A_{11} - S_{11}^T \bar{P}_{11} - S_{12}^T \bar{P}_{21} \\ D_3 &= A_{21} - S_{12}^T P_{11} - S_{22}^T \bar{P}_{21} \end{aligned}$$

We proposed the following algorithm.

$$\begin{aligned} &E_{11}^{(j+1)T} D_0 + D_0^T E_{11}^{(j+1)} = \\ &\quad - V^T H_1^{(j)T} - H_1^{(j)} V + V^T H_3^{(j)} V + \varepsilon H_2^{(j)} \end{aligned} \quad (18a)$$

$$E_{11}^{(j+1)T} D_2 + E_{21}^{(j+1)T} D_4 + D_3^T E_{22}^{(j+1)} = H_1^{(j)} \quad (18b)$$

$$E_{22}^{(j+1)T} D_4 + D_4^T E_{22}^{(j+1)} = H_3^{(j)} \quad (18c)$$

where

$$\begin{aligned} H_1^{(j)} &= -A_{11}^T P_{21}^T + P_{11}^T S_{11} P_{21}^T + P_{21}^T S_{12}^T P_{21}^T \\ &\quad +\varepsilon(E_{11}^{(j)T} S_{12} E_{22}^{(j)} + E_{21}^{(j)T} S_{22} E_{22}^{(j)}) \\ H_2^{(j)} &= E_{11}^{(j)T} S_{11} E_{11}^{(j)} + E_{21}^{(j)T} S_{22} E_{21}^{(j)} \\ &\quad + E_{11}^{(j)T} S_{12} E_{21}^{(j)} + E_{21}^{(j)T} S_{12}^T E_{11}^{(j)} \\ H_3^{(j)} &= -A_{12}^T P_{21}^T - P_{21} A_{12} \\ &\quad +\varepsilon P_{21} S_{11} P_{21}^T + \varepsilon E_{22}^{(j)T} S_{22} E_{22}^{(j)} \\ &\quad + P_{21} S_{12} P_{22} + P_{22}^T S_{12}^T P_{21}^T \end{aligned}$$

4. MAIN RESULTS

The following theorem indicates the convergence features of algorithm (18a)~(18c).

Theorem 1. Under stabilizability-detectability conditions, imposed in Assumption 1 and 2, the algorithm (18a)~(18c) converges to the exact solution of E with the rate of convergence of $O(\varepsilon)$, that is

$$||E - E^{(k)}|| = O(\varepsilon^k), \quad (k = 1, 2, \cdots) \quad (19)$$

or equivalently

$$||E - E^{(k+1)}|| = O(\varepsilon)||E - E^{(k)}|| \quad (20)$$

where

$$E = \begin{bmatrix} E_{11} & E_{21} \\ E_{21}^T & E_{22} \end{bmatrix},\ E^{(k)} = \begin{bmatrix} E_{11}^{(k)} & E_{21}^{(k)} \\ E_{21}^{(k)T} & E_{22}^{(k)} \end{bmatrix}$$

Proof. As a starting point we need to show the existence of a bounded solution of E in neighbourhood of $\varepsilon = 0$. To prove that by the implicit function theorem, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. The Jacobian is given by

$$J|_{\varepsilon=0} = \begin{bmatrix} J_{11} & 0 & 0 \\ J_{21} & J_{22} & J_{23} \\ 0 & 0 & J_{33} \end{bmatrix} \quad (21)$$

where, using the Kronecker products representation we have

$$\begin{aligned} J_{11} &= I \otimes D_0 + I \otimes D_0^T \\ J_{22} &= I \otimes D_4 \\ J_{33} &= I \otimes D_4 + I \otimes D_4^T \end{aligned}$$

The matrix D_4 is nonsingular since Assumption 1 hold. The matix $A_0 - S_0\bar{P}_{11}$ is nonsingular if Assumption 2 hold. Therefore, we obtain the following equation.

$$\begin{aligned} &A_0 - S_0\bar{P}_{11} \\ &= A_{11} + N_1A_{21} + S_{12}N_2^T + N_1S_{22}N_2^T \\ &\quad -(S_{11} + N_1S_{12}^T + S_{12}N_1^T + N_1S_{22}N_1^T)\bar{P}_{11} \\ &= A_{11} + N_1A_{21} - S_{11}\bar{P}_{11} - N_1S_{12}^T\bar{P}_{11} \\ &\quad +S_{12}N_2^T + N_1S_{22}N_2^T \\ &\quad -S_{12}N_1^T\bar{P}_{11} - N_1S_{22}N_1^T\bar{P}_{11} \\ &= A_{11} - S_{11}\bar{P}_{11} + N_1(A_{21} - S_{12}^T\bar{P}_{11}) \\ &\quad -S_{12}(-N_2^T + N_1^T\bar{P}_{11}) \\ &\quad -N_1S_{22}(-N_2^T + N_1^T\bar{P}_{11}) \\ &= A_{11} - S_{11}\bar{P}_{11} - S_{12}\bar{P}_{21} \\ &\quad +N_1(A_{21} - S_{12}^T\bar{P}_{11} - S_{22}\bar{P}_{21}) \\ &= D_1 - D_2D_4^{-T}D_3 = D_0 \end{aligned}$$

The matrix D_0 is stable also. Thus, for ε sufficent small enough the Jacobian is nonsingular. Therefore we can achieve the $O(\varepsilon^k)$ approximation of E by performing only k iteration for algolihtm (18a)~(18c).

5. A NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm (18a)~(18c), we have run a simple example. We consider a nonstandard singularly perturbed systems of the form [5]

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad (22)$$

wiht perfomance index

$$J = \frac{1}{2}\int_0^\infty (x_1^2 + x_2^2 + u^2)dt \qquad (23)$$

The entries show the results obtaind for small parameter $\varepsilon = 0.01$. In the Table 1, the results are presented for the P approximation.

Tab.1. Value of P when $\varepsilon = 0.01$

j	P_{11}	P_{21}	P_{22}
1	1.41414	2.41414	1.0
2	1.43828	2.41414	1.02414
3	1.43799	2.41414	1.02385
4	1.43799	2.41414	1.02386
5	1.43799	2.41414	1.02386
6	1.43799	2.41414	1.02386
7	1.43799	2.41414	1.02386

By using proposed recursive algorithm, we can get the following solutions.

$$u_{pro}^* = -\begin{bmatrix} 2.41414 & 1.02386 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (24)$$

Where, Yue-yun Wang's presented for the exact following solutions.

$$u_{exa}^* = -\begin{bmatrix} 2.4142 & 1.0239 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (25)$$

It can be seen thet the u_{pro}^* converge to exact solution u_{exa}^*.

6. CONCLUSIONS

This paper presented a recusive algorithm for nonstandaed singularly perturbed systems. Using the recursive algorithm, the solution of optimal regulator problem for nonstandard singularly perturbed systems can be obtained with an accuracy $O(\varepsilon^k)$. As a result, the proposed technique represents a significant improvement since the existing method for the standard singularly perturbed systems can not be applied to the nonstandard singularly perturbed systems.

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