CONVERGENCE ANALYSIS OF THE FILTERED-X LMS ACTIVE NOISE CANCELLER FOR A SINUSOIDAL INPUT

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ABSTRACT Application of the filtered-x LMS adaptive filter to active noise cancellation requires to estimate the transfer characteristics between the output and the error signal of the adaptive canceller. We analyze the effects of estimation accuracy on the convergence behavior of the canceller when the input noise is modeled as a sinusoid.

1. INTRODUCTION

In active noise cancellation, the acoustic noise to be cancelled is often generated by rotating machines and thus can be modeled as the sum of a fundamental sinusoid and its harmonics [1]. In this paper we derive an adaptive canceller structure and analyze its convergence behavior when the acoustic noise is assumed a single sinusoid.

2. SYSTEM MODEL

When the noise is a sinusoid, the acoustic and speaker-acoustic-microphone paths [2] can be described by the in-phase (I) and quadrature (Q) weights as shown in the upper branch of Fig. 1. In this case the adaptive canceller structure also becomes to have two weights $w_I(n)$ and $w_Q(n)$, with I and Q inputs, $x_I(n)$ and $x_Q(n)$, respectively. Thus, the output of the canceller, $y(n)$, is expressed as

$$y(n) = w_I(n)\, x_I(n) + w_Q(n)\, x_Q(n) \tag{1}$$

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where $x_I(n) = A\cos(\omega_0 n + \phi_0) \triangleq A\cos\psi_n$ with ω_0 and n being a normalized frequency and discrete time index, respectively. Also, referring to the notation in Fig. 1, the error signal $e(n)$ is represented by

$$e(n) = c_I\, \tilde{e}_I(n) + c_Q\, \tilde{e}_Q(n) + \eta(n) \tag{2}$$

where $\tilde{e}_I(n) \triangleq \tilde{e}(n) = d(n) - y(n)$ and $\tilde{e}_Q(n)$ is the 90° phase-shifted version of $\tilde{e}_I(n)$, and $\eta(n)$ denotes zero-mean measurement noise. Assuming that $w_I(n)$ and $w_Q(n)$ are slowly time-varying as compared to $x_I(n)$ and $x_Q(n)$, the 90° phase-shifted output is given from (1) by

$$y_Q(n) = w_I(n)\, x_Q(n) + w_Q(n)\{-x_I(n)\} \tag{3}$$

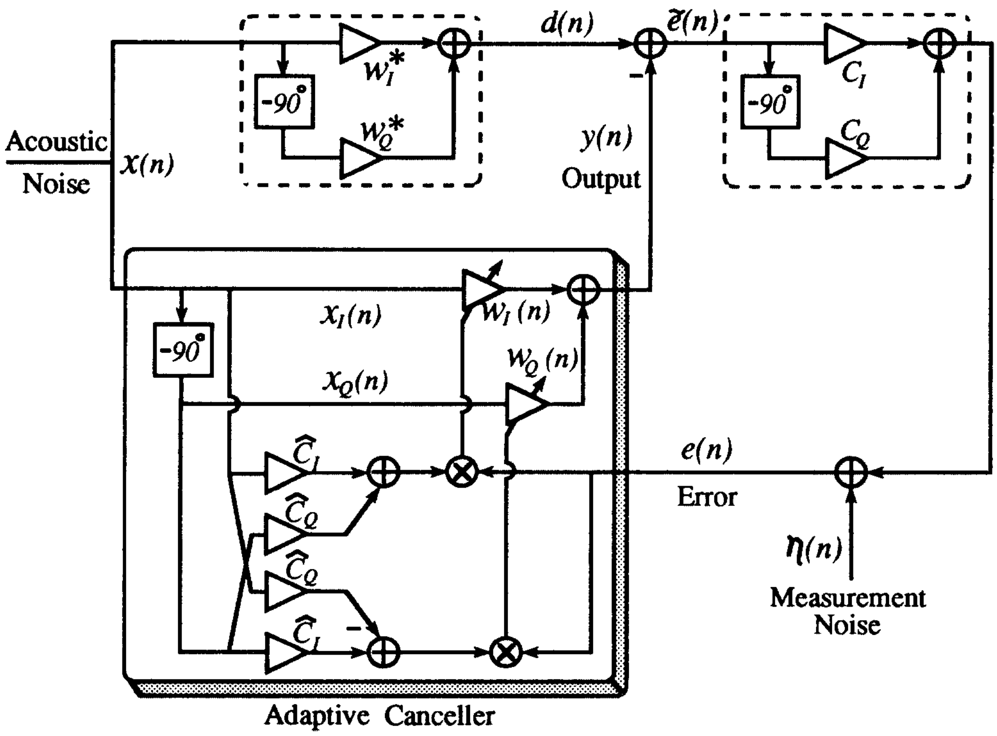

Fig. 1 The diagram of the active noise cancellation system under study.

From (1), (2), and (3), one can obtain an LMS weight update equation by minimizing $e^2(n)$ and using a gradient-descent method as

$$w_I(n+1) = w_I(n) + \mu\, e(n)\{c_I\, x_I(n) + c_Q\, x_Q(n)\}$$

and $\quad w_Q(n+1) = w_Q(n) + \mu\, e(n)\{c_I\, x_Q(n) - c_Q\, x_I(n)\}$ (4)

where μ is a convergence constant. It is noted that to implement the filtered-x LMS algorithm of (4), the values of c_I and c_Q must be estimated [3]. In the following, we analyze the effects of replacing c_I and c_Q in (4) with $\hat{c}_I$ and $\hat{c}_Q$ on the convergence behavior of the canceller.

3. CONVERGENCE ANALYSIS

To see how the adaptive algorithm derived in (4) converges for inaccurate $\hat{c}_I$ and $\hat{c}_Q$, we first investigate the convergence of the expected values of the two adaptive weights. From the underlying signal model (Fig. 1), $E[w_I(n)]$ and $E[w_Q(n)]$ are expected in the steady state to approach w_I^* and w_Q^*, respectively. To simplify the convergence equation, we may introduce two weight errors as

$$v_I(n) \triangleq w_I(n) - w_I^* \quad \text{and} \quad v_Q(n) \triangleq w_Q(n) - w_Q^* \tag{5}$$

Then, from (2), (5) and Fig. 1, we get

$$\tilde{e}_I(n) = -v_I(n)\, x_I(n) - v_Q(n)\, x_Q(n)$$

and $\quad \tilde{e}_Q(n) = -v_I(n)\, x_Q(n) + v_Q(n)\, x_I(n)$ (6)

Inserting (5) into (4) and rearranging the result using (2) and (6), and taking expectation of both sides of the resultant two weight-error equations, we can get the following convergence equation based on the independence assumption on the underlying signals; $x(n)$, $\eta(n)$, $v_I(n)$ and $v_Q(n)$. That is,

$$\begin{bmatrix} E[v_I(n+1)] \\ E[v_Q(n+1)] \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} E[v_I(n)] \\ E[v_Q(n)] \end{bmatrix} \tag{7}$$

where $\alpha \triangleq 1 - \frac{1}{2}\mu A^2 (c_I \hat{c}_I + c_Q \hat{c}_Q)$ and $\beta \triangleq \frac{1}{2}\mu A^2 (\hat{c}_I c_Q - c_I \hat{c}_Q)$.

Here, defining gain and phase response parameters as

$$g \triangleq \sqrt{c_I^2+c_Q^2}\,, \qquad \hat{g} \triangleq \sqrt{\hat{c}_I^{\,2}+\hat{c}_Q^{\,2}}\,,$$

$$\theta_c \triangleq \tan^{-1}\left(\frac{c_Q}{c_I}\right), \qquad \hat{\theta}_c \triangleq \tan^{-1}\left(\frac{\hat{c}_Q}{\hat{c}_I}\right),$$

α and β in (7) can alternatively be expressed as

$$\alpha \triangleq 1-\frac{1}{2}\mu A^2 g\,\hat{g}\cos\Delta\theta_c \quad \text{and} \quad \beta \triangleq \frac{1}{2}\mu A^2 g\,\hat{g}\sin\Delta\theta_c\,. \tag{8}$$

where $\Delta\theta_c \triangleq \theta_c - \hat{\theta}_c$.

Also, using similarity transformation we can convert (7) into the transformed domain as

$$\begin{bmatrix} E[\,\check{v}_I(n+1)] \\ E[\,\check{v}_Q(n+1)] \end{bmatrix} = \begin{bmatrix} 1-\lambda_I & 0 \\ 0 & 1-\lambda_Q \end{bmatrix} \begin{bmatrix} E[\,\check{v}_I(n)] \\ E[\,\check{v}_Q(n)] \end{bmatrix} \tag{9}$$

where $\lambda_i = \frac{1}{2}\mu A^2 g\,\hat{g}\,[\,\cos\Delta\theta_c \pm j\sin\Delta\theta_c\,]$, $\qquad i = I$ and Q.

It should be noted from (9) that since λ_i's are complex values, so are the transformed weight errors. Therefore, we consider the convergence of the magnitude of the transformed error as

$$\rho_i(n+1) = |\,1-\lambda_i\,|\,\rho_i(n), \quad i = I \text{ and } Q \tag{10}$$

where $\rho_i(n) \triangleq |\,E[\check{v}_i(n)]\,|$.

We can see from (10) that the magnitude converges exponentially to zero (*i.e.*, $E[w_i(n)]$ to w_i^*) under the following condition:

$$|\,1-\lambda_i\,| < 1 \qquad \forall_i \tag{11}$$

The time constant of the exponential convergence is derived from the following:

$$e^{-1/\tau_i} \cong 1-\frac{1}{\tau_i} = |\,1-\lambda_i\,| \quad \text{for large } \tau_i,\ i = I \text{ and } Q \tag{12}$$

From (9) and (12) we get

$$\tau_i = \frac{1}{1-\sqrt{1-\mu A^2 g \hat{g} \cos\Delta\theta_c + \frac{1}{4}\mu^2 A^4 g^2 \hat{g}^2}}, \quad i = I \text{ and } Q \tag{13}$$

Next we investigate the convergence of the mean-square-error (MSE), $E[e^2(n)]$. Using (2), (6) and (8) we can express the MSE as

$$E[e^2(n)] = \frac{A^2}{2} g^2 \xi(n) + \sigma_\eta^2 \tag{14}$$

where $\sigma_\eta^2 \triangleq E[\eta^2(n)]$ and $\xi(n) \triangleq E[v_I^2(n)] + E[v_Q^2(n)]$.

It is noted from (14) that the convergence study for the MSE is equivalent to that for the sum of the squared weight errors. Inserting (5), (2) and (6) into (4), squaring and taking expectation of both sides of the result yields

$$\xi(n+1) = \gamma\, \xi(n) + \delta \tag{15}$$

where $\gamma \triangleq 1-\mu A^2 g \hat{g} \cos\Delta\theta_c + \frac{1}{16}\mu^2 A^4 g^2 \hat{g}^2 [9-\cos(2\Delta\theta_c)]$ and

$\delta \triangleq \mu^2 A^2 \hat{g}^2 \sigma_\eta^2$.

Thus, when $|\gamma| < 1$, (15) has the solution as

$$\xi(n) = \gamma^n \xi(0) + \frac{1-\gamma^n}{1-\gamma}\delta \tag{16}$$

Consequently, the convergence of the sum of the squared weight errors can be obtained from (16). The results of the convergence analysis are summarized in Table I.

4. CONCLUDING REMARKS

We can easily see from Table I that the effects of parameter estimation inaccuracy on the convergence behavior of the filtered-x LMS algorithm are characterized by two distinct components : Phase estimation error $\Delta\theta_c$ and estimated magnitude $\hat{g}$. In particular, $|\Delta\theta_c|$ should be less than 90° for convergence. It is, however, noted that once κ_m or κ_v is selected, the convergence turns out to be determined only by $\Delta\theta_c$. The convergence speed is the fastest for $\kappa = 1/2$ regardless of $\Delta\theta_c$. When $\Delta\theta_c = 0$ and $\hat{g} = g$, the convergence result becomes the same as the LMS case. In conclusion, the convergence of the filtered-x LMS algorithm is shown to be strongly affected by the accuracy of the phase response estimate.

Table I The results of the convergence analysis of the Filtered-x LMS algorithm.

	Mean of weight error (Magnitude) $\kappa_m \triangleq \dfrac{\mu A^2 g \hat{g}}{4\cos\Delta\theta_c}$	Summed variance of weight errors $\kappa_v \triangleq \dfrac{\mu A^2 g \hat{g}[9-\cos(2\Delta\theta_c)]}{16\cos(\Delta\theta_c)}$
Stability condition	$0<\mu<\dfrac{4\cos\Delta\theta_c}{A^2 g \hat{g}}$ or $0 < \kappa_m < 1$	$0<\mu<\dfrac{16\cos\Delta\theta_c}{A^2 g \hat{g}[9-\cos(2\Delta\theta_c)]}$ or $0 < \kappa_v < 1$
Time constant	$\dfrac{1}{1-\sqrt{1-4\kappa_m(1-\kappa_m)\cos^2\Delta\theta_c}}$	$\dfrac{9-\cos(2\Delta\theta_c)}{16\kappa_v(1-\kappa_v)\cos^2\Delta\theta_c}$
Steady-state value	0	$\dfrac{16\kappa_v\sigma_\eta^2}{A^2 g^2(1-\kappa_v)[9-\cos(2\Delta\theta_c)]}$

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