Application of a Rule-based Expert System to the Postoptimality Analysis for the Linear Goal Programming Model

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ABSTRACT

Goal programming has been recognized as an effective tool for many real world problems, and the analytical procedures for postoptimality analysis have already been developed. Heuristic measures to reflect the decision maker's preference and to support consistent judgment are considered for the cases of alternate optima and unimplementable solutions in this research.

In the linear goal programming algorithm, an unimplementable solution occurs when the problem has no solution that satisfies the rigid constraints of priority 1. This case is very similar to that of an infeasible solution for the single-objective linear programming problem. A problem has an alternative optimal set of solutions if the solution space associated with that problem has more than a single point. Furthermore, given that the solution space is a region, any point in that region or any boundary of that region is an optimal solution. For above two cases of solutions, decision maker may prefer a specific solution subjectively. An expert system is chosen as a tool to implement several heuristic rules and to assist the decision maker's decision.

With the help of special structuring of the linear goal programming model, those problems that could not be considered previously can be settled in postoptimality analysis.

1. Introduction

In most practical situations, some of the model coefficients are not known exactly and hence are estimated as well as possible. Also, processes may vary and change with time, or confidence in either the priority structure or the intrapriority weights may be low. It is desirable to examine the effect of relaxing some of the constraints on the values of optimal objectives without resolving the entire problem.

Besides the general meaning of postoptimality analysis mentioned above, proper measures need to be devised for unimplementable and alternate optimal solutions. For example, when the simplex algorithm results in an infeasible solution, the only option is to return to the starting point and thoroughly check the mathematical model, since an infeasible solution is caused by incorrect model building due to an incorrect transformation of reality. When we utilize the special structure of goal programming (GP), aforementioned problems could be solved.

A system which can manipulate more efficiently for those solutions is developed when applying GP model and algorithm. Unimplementable solutions could be corrected by changing a part of the decision maker's model, and this correction can be discussed with the decision maker (DM) and guided by our system. For alternative optimal solutions, our system makes the DM freely select his or her preferred solution within the optimal range of continuous line.

2 Case of Unimplementable Solution

In the GP algorithm, an unimplementable solution occurs when the problem has no solution that satisfies the rigid constraints of priority 1, and is detected by a positive value of $A^{(1)}$. The case is very similar to that of an infeasible solution for the single-objective linear programming problem in the sense that there is no single point which can satisfy all rigid constraints. Since the simplex algorithm cannot go beyond the feasible region, no further implementation is possible and the algorithm terminates in the LP problem. However, the GP algorithm can possibly indicate the solution that is nearest to being implementable, and it can be determined which rigid constraint(s) must be relaxed if an implementable solution is to be obtained.

The first achievement vector is the summation of unwanted deviation(s) for each absolute goal, i.e., rigid constraint, which should be satisfied in order to be at least implementable. From the above fact and the definition of an unimplementable solution

$(A^{(1)} > 0)$, it can easily be recognized that one or more unwanted deviation variables are in the set of basic variables and the variables take on positive values.

When this case occurs, our system discontinues the implementation of the GP algorithm, and starts the routine for correcting the right-hand-side element of the violated constraint. The RHS correcting routine for the unimplementable solution case can be summarized as follows:

Let i and β denote row number and value of the violated deviation variable.

Step 1. Detect a deviation variable which takes on positive value (η_i or ρ_i),

If there is no more positive valued deviation variable, go to Step 6,

Step 2. Find violating rigid constraint from i,

Step 3. If $\eta_i > 0$, ask the DM if decreasing more than β units in i-th RHS is possible,

If $\rho_i > 0$, ask the DM if increasing more than β units in i-th RHS is possible,

If the DM's response is negative, then go to Step 5,

Otherwise, go to step 5,

Step 4. Receive the DM's input value,

If $\eta_i > 0$, subtract input amount from the original RHS value,

If $\rho_i > 0$, add input amount to the original RHS value, go to Step 1,

Step 5. No further analysis is possible. Routine ends, and

Step 6. Perform rerun routine.

3. Case of Alternative Optimal Solutions

A problem has an alternative optimal set of solutions if the solution space associated with that problem has more than a single point. Furthermore, given that the solution space is a region, any point in that region or any boundary of that region is an alternative optimal solution. Any solution in the solution space will produce identical achievement level vector A^* and different variable values. The existence of alternative optimal solutions is indicated by an entire column of zero-valued reduced costs $R_N^{(k)}$ ($k = 1, 2, \cdots, K$ and N: set of nonbasic variables) in the final multiphase tableau.

Let x_j denote the nonbasic variable with zero valued $R_j^{(k)}$ for all levels. When x_j enters into the basis, changes in the achievement level vector can be proven to be zero mathematically by

$$A_{new}^{(k)} = c_B^{(k)} B^{-1} \bar{r} - R_j^{(k)} x_j \quad k = 1, 2, \cdots, K \qquad (1)$$

Since $R_j^{(k)}$ for all priority levels are zeroes, the achievement level vector will not be changed by entering x_j into the basis. The change of the variable value may be changed as follows:

$$X_B^{new} = B^{-1}\bar{r} - y_j x_j . \qquad (2)$$

The minimum ratio rule is applied. Let α and i denote the minimum positive value after the test and the row that the ratio is α. Then the basic variable x_{B_i} leaves the basis, the value of x_j is increased to α and the alternative optimal variable values are changed by Equation (2) when x_j enters into the basis. A simple example should make this clear and we will explain how our system works at this point with an example problem. An example GP model which has alternative optimal solutions is given as follows:

$$\textit{lexico. } \min. \ \{(\rho_1 + \rho_2), (-2x_1 - 4x_2)\}, \qquad (3)$$

subject to

$$x_1 + 2x_2 + \eta_1 - \rho_1 = 4, \qquad (4)$$
$$-x_1 + x_2 + \eta_2 - \rho_2 = 1, \text{ and} \qquad (5)$$
$$x_i, \ \eta_i, \ \rho_i \geq 0 \quad i = 1,2. \qquad (6)$$

Table 1 The Final Multiphase Tableau for Alternate Optima.

		$c^{(2)}$	-4	0	0	0	
		$c^{(1)}$	0	0	1	1	
P_2	P_1		x_2	η_1	ρ_1	ρ_2	RHS
-2	0	x_1	2	1	-1	0	4
0	0	η_2	3	1	-1	-1	5
		$R_N^{(1)}$	0	0	-1	-1	
		$R_N^{(2)}$	0	-2	2	0	

Table 1 shows the final multiphase tableau. As we can observe at this optimal tableau, the reduced cost rows of the x_2 column are zeroes, which means that alternate optima exist. Optimal achievement level vectors remain unchanged from Equation (1). x_2 can be allowed to enter the basis while maintaining the optimal condition. The changes in the basic variables and x_2 can be represented as

$$\begin{bmatrix} x_1 \\ \eta_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4-2x_2 \\ 5-3x_2 \\ x_2 \end{bmatrix}. \qquad (7)$$

The maximum value which can satisfy the nonnegativity condition for each variable (minimum ratio rule) is $5/3$.

If the value of x_2 increases to a value greater than $5/3$, the value of η_2 negative. When the value of x_2 is equal to $5/3$, $[x_1 \quad \eta_2] = [2/3 \quad 0]$; thus, the zero-valued η_2 leaves the basis.

At this point, it must be noted that optimal solutions are not just the two points that correspond to $[x_1 \quad x_2 \quad \eta_2] = [4 \quad 0 \quad 5]$ and $[2/3 \quad 5/3 \quad 0]$, but any point on the line segment joining these two points. By selecting a value for x_2 $(0 \le x_2 \le 5/3)$, values of x_1 and η_2 can be easily computed. This situation can now be generalized. Let p_1 and p_2 denote the extreme points that have like achievement levels at optimality. Then, any point which is a convex combination of p_1 and p_2 is also an optimal solution. It can be represented mathematically as

$$\lambda p_1 + (1-\lambda) p_2 , \quad \text{where } \lambda \in [0,1]. \qquad (8)$$

Unfortunately, the traditional method cannot find middle points $(0 < \lambda < 1)$, because the simplex procedure cannot allow an optimal solution which is not an extreme point. The number of basic variables should be equal to the number of constraints, and the number of positive-valued variables cannot exceed the number of basic variables. From the above example, if $x_2 = 1$, then $x_1 = 2$ and $\eta_2 = 2$. Only two variables can be in the basis, and there are three positive-valued variables. It contradicts the basic assumption of the simplex procedure that there are (number of variables - number of constraints) nonbasic variables with zero values.

In the case of alternative optima, the focal point is the fact that the DM may prefer a middle point optimal solution. An iterative consulting routine was developed to obtain the DM's most desirable optimal solution, and can be summarized as follows:

Step 1. Show the existence of alternate optima.
Step 2. Compute the minimum ratio for a nonbasic variable with all zero reduced costs.
Step 3. Show the possible range and receive the DM's preferred value for the nonbasic variable.
Step 4. If the input value is out of range, go to Step 3.
Otherwise, show the optimal result.
Step 5. Ask DM whether the result is satisfied or not.
If yes, append the DM's preferred solution to the output summary. Routine ends.
If no, go to Step 3.

This iterative routine was developed to prevent unexpected results which can degrade a DM's satisfaction. Sensitivity analysis in the general case cannot be continued with non-extreme point solutions. This problem could be solved by adding a constraint. Let δ denote the value for the nonbasic variable x_j which is determined by the DM after implementing the iterative routine from above. The sensitivity analysis will be started from the original solution $(\delta = 0)$. our system provides the routine for adding the new constraint. Select this routine from the menu and add the constraint $x_j = \delta$. By doing this, the DM can obtain his or her preferred solution and perform other types of sensitivity analysis. User interaction during postoptimality analysis is the key reason for implementing our system.

4. Conclusions and Further Research

In this research, a rule based expert system is applied to implement several heuristic rules and to reflect the DM's preference for unimplementable and alternative optimal solutions.

For the case of alternative optimal solutions, the nonextreme point optimal solution is allowed by our system when the DM so prefers. For the case of unimplementable solution, the system provides routines which enable those cases to be converted to implementable solutions by changing a right-hand-side value in the constraint set or by adding a constraint. Consequently, this research have been focused on the improvement of system efficiency, the direct reflection of the DM's preferences, and the improvement of user-friendliness.

For the case of unimplementable solution, changes of the RHS value are manipulated to obtain implementable solution. The other measure which could solve the problem by correcting the technical coefficients is desirable to develop for further research. Alternate optimal solution module needs to be generalized further. Expert systems may serve as a good tool for developing such procedures.

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