Service Stations with Batch Arrivals and Batch Services

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ABSTRACT

A service system with batch arrivals and batch services, represented by $M_n^b/M_n^b/1$, is considered. This is a single-server system in which batches of units arrive according to a Poisson process with rate depending on the number in the system and the batch sizes are i.i.d. and have a geometric distribution. Similarly, the service process is a batch service system in which batches depart according to a Poisson process with rate depending on the number in the system and the batch sizes are independent truncated geometric variables. We derive the closed-form equilibrium distribution of the number of units in this $M_n^b/M_n^b/1$ system.

1. Introduction

A service system with batch arrivals and batch services, represented by $M_n^b/M_n^b/1$, is considered. This is a single-server system in which batches of units arrive according to a Poisson process with rate depending on the number in the system and the batch sizes are i.i.d. and have a geometric distribution. The neumonic M_n^b refers to this type of process. Similarly, the service process, also represented by M_n^b, is a batch service system in which batches depart according to a Poisson process with rate depending on the number in the system and the batch sizes are independent truncated geometric variables. We derive the equilibrium distribution of the number of units in this $M_n^b/M_n^b/1$ system. One can interpret the $M_n^b/M_n^b/1$ system as a generalized birth and death process where the births and deaths occur in batches or groups. A special case is the $M^b/M/1$ queueing system(M^b means compound Poisson with geometric batches). Other special cases of the $M_n^b/M_n^b/1$ system, which have not been studied before, are the systems $M_n^b/M/s$, $M_n^b/M/\infty$, $M_n^b/M_n/1$, $M_n/M_n^b/1$, etc. (M_n means state-dependent Poisson Process).

There are several related studies of batch service queues. Bagchi and Templeton[1] studied the $M^b/M^b/1/K$ queueing system which allows at most K units in the system. They provided a numerical method for evaluating the queue size distribution. Chiamsiri and Leonard[2] introduced a diffusion approximation for the equilibrium distribution of the number of units in the single server system with batch-arrival and batch-service. Powell and Humblet[3] provided a computational procedure for the queue length distribution and for the moments of the queue length distribution for certain batch arrival and batch service queues. None of these studies give closed-form equilibrium distributions.

2. $M_n^b/M_n^b/1$ Queue

Consider a service system in which units arrive in batches and are served in batches as follows. When there are n units in the system, the time to the next potential batch arrival is exponentially distributed with rate λ_n, and the batch size has the geometric distribution $(1-\alpha)\alpha^{k-1}$, $k=1,2,\cdots$. Also the time to the next potential departure of a batch is exponentially distributed with rate μ_n. The number of units that may depart in a batch has the geometric distribution $(1-\beta)\beta^{k-1}$, $k=1,2,\cdots$, and so when n units are present the actual number that departs in a batch has the truncated geometric distribution

$$g_n(k)=\begin{cases}(1-\beta)\beta^{k-1} & k=1,\cdots,n-1\\ \beta^{n-1} & k=n.\end{cases}$$

Let the process $\{X(t):t\geq 0\}$ represent the number of units in the system at any time. Under the preceding assumptions, it follows that X is a Markov process with transition rates

$$\begin{aligned} q(n,n+k) &= \lambda_n(1-\alpha)\alpha^{k-1} \\ q(n,n-k) &= \mu_n g_n(k) \qquad k=1,\cdots n. \end{aligned}$$

To describe the equilibrium distribution, we will use the product

$$r_n = \prod_{k=1}^{n}\frac{\lambda_{k-1}+\alpha\mu_{k-1}}{\beta\lambda_k+\mu_k}$$

where $\mu_0=0$.

Theorem 1 If $\sum_{n=1}^{\infty} r_n < \infty$, then the equilibrium distribution of the $M_n^b/M_n^b/1$ process X is

$$\begin{aligned} p_0 &= [(1-\alpha\beta)\sum_{n=1}^{\infty}r_n]^{-1} \\ p_n &= p_0 r_n(1-\alpha\beta) \qquad n=1,2,\cdots. \end{aligned} \tag{1}$$

Our proof of this will use the following property of the p_n.

Lemma 1 The probabilities p_n in Theorem 1 satisfy

$$\sum_{k=0}^{n-1}\lambda_k p_k\alpha^{n-1-k}=\frac{(\beta\lambda_n+\mu_n)p_n}{1-\alpha\beta} \qquad n=1,2,\cdots.$$

Proof : For n=1, the expression is

$$\lambda_0 p_0=\frac{(\beta\lambda_1+\mu_1)p_1}{1-\alpha\beta}$$

which is equivalent to $p_1=p_0r_1(1-\alpha\beta)$, and this is true by (1). Proceeding by induction, assume the assertion is true for n. Then

$$\begin{aligned} \sum_{k=0}^{n}\lambda_k p_k\alpha^{n-k} &= \lambda_n p_n+\alpha\frac{(\beta\lambda_n+\mu_n)p_n}{1-\alpha\beta} \\ &= \frac{p_n}{1-\alpha\beta}(\lambda_n+\alpha\mu_n) \\ &= \frac{(\beta\lambda_{n+1}+\mu_{n+1})p_{n+1}}{1-\alpha\beta} \end{aligned}$$

The last line follows since

$$p_{n+1}=p_n\frac{\lambda_n+\alpha\mu_n}{\beta\lambda_{n+1}+\mu_{n+1}}.$$

Thus the induction is complete.

Proof of Theorem 1 : The balace equations for the process X are

$$\lambda_0 p_0 = \sum_{k=1}^{\infty} p_k\mu_k\beta^{k-1}$$

$$(\lambda_n+\mu_n)p_n = (1-\beta)\sum_{k=1}^{\infty} p_{n+k}\mu_{n+k}\beta^{k-1}+(1-\alpha)\sum_{k=1}^{n} p_{n-k}\lambda_{n-k}\alpha^{k-1} \quad n\geq 1.$$

Define

$$S_n=\sum_{k=n+1}^{\infty} p_k\mu_k\beta^{k-1}.$$

The balance equations can be expressed as

$$\lambda_0 p_0 = S_0 \tag{2}$$

$$(\lambda_n+\mu_n)p_n = (1-\beta)\beta^{-n}S_n+(1-\alpha)\sum_{k=0}^{n-1} p_k\lambda_k\alpha^{n-k-1} \quad n\geq 1.$$

We will now prove (1) by induction. To prove this for n=1, consider the first two balance equations

$$\lambda_0 p_0 = S_0$$

$$(\lambda_1+\mu_1)p_1 = (1-\beta)\beta^{-1}(S_0-p_1\mu_1)+(1-\alpha)S_0$$

Solving these for p_1 yields $p_1=p_0r_1(1-\alpha\beta)$. Thus (1) is true for n=1. Now assume (1) is true for some n. The (n+1)st balance equation can be written as

$$(\lambda_{n+1}+\mu_{n+1})p_{n+1} = (1-\beta)\beta^{-n-1}(S_n-p_{n+1}\mu_{n+1}\beta^n)+(1-\alpha)\sum_{k=0}^{n} p_k\lambda_k\alpha^{n-k} \tag{3}$$

since $S_{n+1}=S_n-p_{n+1}\mu_{n+1}\beta^n$. From the nth balance equation we know that

$$(1-\beta)\beta^{-n}S_n = (\lambda_n+\mu_n)p_n-(1-\alpha)\sum_{k=0}^{n-1} p_k\lambda_k\alpha^{n-k-1}.$$

Substituting this in (3) and arranging terms, we have

$$[\lambda_{n+1}+\mu_{n+1}+\beta^{-1}(1-\beta)\mu_{n+1}]p_{n+1}$$
$$=\beta^{-1}(\lambda_n+\mu_n)p_n+[\alpha(1-\alpha)-\beta^{-1}(1-\alpha)]\sum_{k=0}^{n-1} p_k\lambda_k\alpha^{n-k-1}+(1-\alpha)p_n\lambda_n$$

Multiplying this by β and applying Lemma 1 to the last sum, we get

$$(\beta\lambda_{n+1}+\mu_{n+1})p_{n+1}$$
$$=(\lambda_n+\mu_n)p_n+[\alpha(1-\alpha)\beta-(1-\alpha)]\frac{(\beta\lambda_n+\mu_n)p_n}{1-\alpha\beta}+(1-\alpha)\beta p_n\lambda_n$$

Solving this for p_n yields

$$p_{n+1}=p_n\frac{\lambda_n+\alpha\mu_n}{\beta\lambda_{n+1}+\mu_{n+1}}.$$

Thus the induction is complete.

Comments on the Proof Although the preceding proof is short, our initial derivation of p_n was rather lengthy. We first conjectured the form of the equilibrium distribution of the $M_n^b/M_n^b/1$ system by following steps. The $M/M/1$ system has the equilibrium distribution

$$p_n = p_0\left(\frac{\lambda}{\mu}\right)^n.$$

and the $M_n/M_n/1$ system has the equilibrium distribution

$$p_n = p_0\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}.$$

We applied the same logic to the $M^b/M/1$ system to guess the form of the equilibrium distribution of the $M_n^b/M_n/1$ system, which is shown on line 2 of Table 1. Next, we noted that if $\lambda_0=\alpha\mu+\lambda$, $\lambda_n=\lambda$ for all $n\geq1$ and $\mu_n=\mu$ for all $n\geq1$, then the $M_n^b/M_n/1$ system has the equilibrium distribution

$$p_n = p_0\left(\frac{\alpha\mu+\lambda}{\mu}\right)^n \tag{4}$$

By considering the reversed process, we could get the equilibrium distribution of the $M/M^b/1$ system (line 3 of Table 1). From this, we could guess the equilibrium distribution of $M_n/M_n^b/1$ system (line 4 of Table 1). Next, we noted that both the $M/M/1$ and $M/M^b/1$ systems have equilibrium distributions of the form

$$p_n = p_0 c^n$$

for some constant c. Similarly, from (4) we guessed that the form of the equilibrium distribution of the $M_n^b/M_n^b/1$ system would be

$$p_n = p_0 c^n$$

if $\lambda_n=\lambda$ for all $n\geq1$ and $\mu_n=\mu$ for all $n\geq1$ and λ_0 is chosen appropriately. By considering the reversed process, we could get the equilibrium distribution of the $M^b/M^b/1$ system (line 5 of Table 1). From this we could guess the equilibrium distribution of the $M_n^b/M_n^b/1$ system (line 6 of Table 1). Even this involved a few wrong guesses. Eventually, we found the distribution and verified it by a long substitution arguments. Finally, we arrived at the shorter induction proof.

Comments on Examples of the $M_n^b/M_n^b/1$ System

(a) As we mentioned in the introduction, this process X can be interpreted as a generalized birth and death process where the births and deaths occur in groups.

(b) The M_n^b service process is applicable for modeling "dispatching" or "instantaneous services" in the following sence. A truck (or computer bus, AGV, dispatcher, etc.) arrives to the station periodically at times that form a Poisson process with rate μ_n. Upon arriving, the truck instantaneously takes or dispatches a batch of units from the station. Due to other environmental influences, the capacity of the truck may have the geometric distribution $(1-\beta)\beta^{k-1}, k=1,2,\ldots$. Then the actual number of units that depart when n are present is the truncated geometric distribution g_n.

(c) Although the $M_n^b/M_n^b/1$ system refers to 'one' server, it also represent multiple server systems. For instance, $M_n^b/M/s=M_n^b/M_n/1$ with $\mu_n=\mu\min\{n,s\}$.

(d) The classical machine repair system [4] has an obvious extension to batch repair in which the number of machines in repair would be an $M_n/M_n^b/1$ process with $\lambda_n=(N-n)\lambda$, where N is the number of machines each with failure rate λ.

Table 1: Table of Equilibrium Distributions fro Batch Arrival and Batch Service Queueing System

System	Transition Rates	Equilibrium distribution
$M^b/M/1$	$q(n,n+k) = \lambda(1-\alpha)\alpha^{k-1}$ $q(n,n-1) = \mu$	$p_n = p_0 \frac{\lambda}{\mu}(\frac{\alpha\mu+\lambda}{\mu})^{n-1}$
$M_n^b/M_n/1$	$q(n,n+k) = \lambda_n(1-\alpha)\alpha^{k-1}$ $q(n,n-1) = \mu_n$	$p_n = p_0 \prod_{k=1}^{n} \frac{\alpha\mu_{k-1}+\lambda_{k-1}}{\mu_k}$
$M/M^b/1$	$q(n,n+1) = \lambda$ $q(n,n-k) = \{ \begin{matrix} \mu(1-\beta)\beta^{k-1} & *a \\ \mu\beta^{n-1} & *b \end{matrix}$	$p_n = p_0 (\frac{\lambda}{\beta\lambda+\mu})^n$
$M_n/M_n^b/1$	$q(n,n+1) = \lambda_n$ $q(n,n-k) = \{ \begin{matrix} \mu_n(1-\beta)\beta^{k-1} \\ \mu_n\beta^{n-1} \end{matrix}$	$p_n = p_0 \prod_{k=1}^{n} \frac{\lambda_{k-1}}{\beta\lambda_k+\mu_k}$
$M^b/M^b/1$	$q(n,n+k) = \lambda(1-\alpha)\alpha^{k-1}$ $q(n,n-k) = \{ \begin{matrix} \mu(1-\beta)\beta^{k-1} \\ \mu\beta^{n-1} \end{matrix}$	$p_n = p_0 \frac{\lambda(1-\alpha\beta)}{\beta\lambda+\mu}(\frac{\alpha\mu+\lambda}{\beta\lambda+\mu})^{n-1}$
$M_n^b/M_n^b/1$	$q(n,n+k) = \lambda_n(1-\alpha)\alpha^{k-1}$ $q(n,n-k) = \{ \begin{matrix} \mu_n(1-\beta)\beta^{k-1} \\ \mu\beta^{n-1} \end{matrix}$	$p_n = p_0 (1-\alpha\beta) \prod_{k=1}^{n} \frac{\alpha\mu_{k-1}+\lambda_{k-1}}{\beta\lambda_k+\mu_k}$

a: if $1 \leq k < n$
b: if $k = n$

References

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[4] Gross, D. and Harris, C. M., "Fundamentals of Queueing Theory", John Wiley and Sons, 1985.