On the Complexity of the Production-Transportation Problem

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Abstract

The Production-Transportation problem, (*PTP*) is a generalization of the transportation problem. In (*PTP*), we decide not only the level of shipment from each source to each sink but also the level of supply at each source. A concave production cost function is associated with the assignment of supplies to sources. Thus the objective function of (*PTP*) is the sum of the linear transportation cost and the production costs. We show that this problem in general is *NP*-hard and present some polynomial classes. Especially, the problem is polynomial when the transportation cost matrix has the Monge property and the number of sources is fixed. The algorithm generalizes a polynomial algorithm of [TDG92] for the problem with two sources.

Keywords: Production-Transportation Problem, concave minimization, parametric linear programming, Monge sequence

It is known that the minimization problem over a polyhedron is polynomial when the objective function is *convex* [GLS88]. In contrast, many concave minimization problems are *NP*-hard. We consider here a concave minimization problem over transportation constraints called the *Production-Transportation* problem, (*PTP*). (*PTP*) is a generalization of the transportation problem. In (*PTP*), we need to decide not only the level of shipment from each source to each sink but also the level of supply at each source. A concave production cost function is associated with the assignment of supplies to sources. The objective function is the sum of the linear transportation costs and the concave production costs.

A special class of this problem has been previously studied by Tuy, Dan and Ghannadan [TDG92]. It was shown that when there are only two sources, the problem can be reduced to a problem of finding all breakpoints of a parametric two-source linear transportation problem with a parametrized supply level. The number of breakpoints of the parametric problem is bounded by the number of sinks and can be found in strongly polynomial time. For a two-source problem, this results in a strongly polynomial time algorithm.

We prove here that (*PTP*) is *NP*-hard and describe subclasses which are solvable in polynomial time. One polynomial subclass is (*PTP*) with fixed number of sinks. Another polynomial subclass is the problem with fixed number of sources and a transportation cost matrix satisfying the *Monge* property [Hof63]. The Monge property of a matrix is recognizable in polynomial time [ACHS89]. Trivial examples of matrices with Monge property are those with

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identical costs in the rows or columns of the transportation cost matrix. With such a matrix the problem is solvable in linear time even with an arbitrary number of sources or sinks.

The paper is organized as follows. Section 1 presents the formulation of (*PTP*). In Section 2, we prove that (*PTP*) is *NP*-hard and discuss some polynomial subclasses of (*PTP*). Section 3 presents a polynomial algorithm for the problem with a transportation cost matrix satisfying Monge property and a fixed number of sources. Finally, some open problems are presented in Section 4.

1. The Production-Transportation Problem

Consider a transportation problem with a set of sources, $\{1,\ldots,m\}$ and a set of sinks, $\{1,\ldots,n\}$. Let c_{ij} be the cost of transporting a unit from i to j. The supply allocation to the sources are not prescribed but determined subject to a concave function $g(x_1,x_2,\ldots,x_m)$ for $x_1, x_2,\ldots, x_m$, the variables representing the supply levels at the m sources. The problem is to allocate the supplies to the sources and to send them to the sinks at the minimum total cost. Thus the Production-Transportation problem,(*PTP*), which is formulated as follows, is a generalization of the transportation problem:

$$(PTP) \qquad \min g(x_1,x_2,\ldots,x_m) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$
$$\sum_{j=1}^{n} x_{ij} = x_i \quad i=1,\ldots,m$$
$$\sum_{i=1}^{m} x_{ij} = b_j \quad j=1,\ldots,n$$
$$x_i, x_{ij} \ge 0 \quad i=1,\ldots,m,\ j=1,\ldots,n.$$

It is reasonable to assume that the marginal cost of production decreases as the production level increases. The production cost function g which is concave reflects this cost economies of scale. Note that if the production system is homogeneous for all supply centers, the cost function g is symmetric.

Let $B=b_1+b_2+\cdots+b_n$ be the total demand. (*PTP*) is feasible if and only if $x_1+x_2+\cdots+x_m=B$ and each x_i is nonnegative. A nonnegative vector, $x\in \mathbf{R}^m$ is called a *feasible production plan* if $x_1+x_2+\cdots+x_m=B$.

Let $\mathbf{x}^1,\mathbf{x}^2\in \mathbf{R}^m$ be feasible production plans. Consider the production plan, $\mathbf{x}^3=\frac{1}{2}(\mathbf{x}^1+\mathbf{x}^2)$. Then $\mathbf{x}^3$ is also a feasible production plan. Note that the concavity of g implies that the cost of the production plan $\mathbf{x}^3$ is at least as much as the sum of the halves of the costs of the production plan $\mathbf{x}^1$ and $\mathbf{x}^2$.

It was shown by Tuy et al, [TDG92] that when $m=2$, (*PTP*) is solvable in $O(n \log n)$ elementary operations assuming an oracle for function evaluations which provides a single evaluation as unit operation. In subsequent sections, we show that (*PTP*) is *NP*-hard and present some polynomial subclasses.

2. The Complexity Status of (*PTP*)

2.1 (*PTP*) is NP-hard

We are unable to argue the membership of (the decision) problem of (*PTP*) in *NP* since g is assumed to be an arbitrary concave function and the input size of g is not well-defined. We assume a computation model with a function evaluation oracle providing a single evaluation as unit operation.

We demonstrate the NP-hardness of (PTP) by showing that (PTP) is at least as hard as a known NP-hard problem - the optimization version of a known NP-complete decision problem. As g may be arbitrary, the polynomial reducibility of a NP-hard problem to (PTP) needs to be elaborated upon. In this paper, we say that an NP-hard problem, (P), is polynomially reducible to (PTP) if,

i) (P) is formulated as an instance of (PTP), and

ii) the sum of the input sizes of the numbers in the instance (excluding the concave function g) is polynomially bounded in the input size of (P).

According to this definition, we prove that (PTP) is NP-hard by showing that the optimization problem **SET COVER** (See **MINIMUM COVER** of [GJ79]) is polynomially reducible to (PTP).

SET COVER
Given a collection $K=\{S_1,S_2,\ldots,S_m\}$ of subsets of a finite set $S=\{1,2,\ldots,n\}$. Find a subcollection K' of K of a minimum number of subsets such that the union of the subsets in K' is equal to S.

For a minimum cover K' of S, define a map from S to K' which associates a set $S(j)\in K'$ with each element j of S so that $j\in S(j)$. Such map exists since K' is a cover. Also the map is onto since otherwise the cover is not minimum.

SET COVER can be viewed as the problem of associating with each element $j\in S$ a subset $S(j)$ containing j so that the total number of the subsets utilized in the association is minimum.

Theorem 2.1: (PTP) is NP-hard.

Proof: We show that **SET COVER** is polynomially reducible to (PTP).

For $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$, let

$$x_{ij}=\begin{cases}1 & \text{if } S_i=S(j)\\ 0 & \text{otherwise.}\end{cases}$$

Since each element is associated with exactly one set, $\sum_{i=1}^{m} x_{ij}=1$ for each $j=1,\ldots,n$.

Consider (PTP) with transportation costs

$$c_{ij}=\begin{cases}0 & \text{if } j\in S_i\\ M & \text{otherwise,}\end{cases}$$

for M a constant greater than $1/n$ and a concave production cost function,

$$g(x_1,x_2,\ldots,x_n)=(x_1+1)^{1/n}+(x_2+1)^{1/n}+\cdots+(x_n+1)^{1/n}.$$

The (PTP) problem formulation with these costs will be shown to solve **SET COVER**:

$$(P2.1)\qquad \min\ (x_1+1)^{1/n}+(x_2+1)^{1/n}+\cdots+(x_n+1)^{1/n}+\sum_{\substack{i,j \text{ s.t.}\\ j \text{ nomem } S_i}} M\,x_{ij}$$

$$\sum_{j=1}^{n} x_{ij}=x_i \quad i=1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij}=1 \quad j=1,\ldots,n$$

$$x_{ij}\ge 0 \quad i=1,\ldots,m,\ j=1,\ldots,n.$$

Due to the concavity of the objective function an optimal solution of $(P2.1)$ is attained at a vertex where each x_{ij} is 0 or 1. Furthermore, we show that given any optimal solution, an optimal basic feasible solution can be constructed in strongly polynomial time. It follows that the integrality constraints for x_{ij} may be omitted in $(P2.1)$.

Let $\bar{\mathbf{x}}=(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_m;\bar{x}_{11},\bar{x}_{12},\ldots,\bar{x}_{mn})$ be an optimal solution of $(P2.1)$.

Claim 1: $\bar{x}_{ij}>0$ only if $c_{ij}=0$.

Proof: Assume the contrary: there exists a pair $1\le p\le m$ and $1\le q\le n$ such that $\bar{x}_{pq}>0$ and $c_{pq}=M$. Since **SET COVER** is feasible, there is $1\le p'\le m$ such that $c_{p'q}=0$. Consider the solution $\hat{\mathbf{x}}$ obtained from $\bar{\mathbf{x}}$ by modifying the values of x_{pq} and $x_{p'q}$ as follows:

$$x_{pq}\leftarrow 0 \text{ and } x_{p'q}\leftarrow \bar{x}_{p'q}+\bar{x}_{pq}.$$

Since $x_{p'q}$ is increased by $\bar{x}_{pq}$, $x_{p'}$ is also increased by $\bar{x}_{pq}$. Since $(x_{p'}+1)^{1/n}$ is strictly concave, the increase in the objective function due to the increase in $x_{p'q}$ is less than,

$$\frac{1}{n}(\bar{x}_{p'q}+1)^{\frac{1-n}{n}}\bar{x}_{pq}\le\frac{1}{n}\bar{x}_{pq},$$

where $\frac{1}{n}(\bar{x}_{p'}+1)^{\frac{1-n}{n}}$ is the derivative of $(x_{p'}+1)^{1/n}$ at $x_{p'}=\bar{x}_{p'}$.

On the other hand, the decrease in the objective function due to the decreasing of $\bar{x}_{pq}$ is greater than $M\,\bar{x}_{pq}$. Thus the net change of the objective value is less than $(\frac{1}{n}-M)\bar{x}_{pq}$.

Since $M>1/n$ the new solution $\hat{\mathbf{x}}$ has objective function value less than that of $\bar{\mathbf{x}}$. This contradicts the optimality of $\bar{\mathbf{x}}$ for $(P2.1)$. Thus we conclude that $\bar{x}_{ij}>0$ only if $c_{ij}=0$. □

Claim 2: $\bar{x}_i$ is integer for every $i=1,\ldots,m$.

Proof: Suppose there is $1\le r\le m$ such that $\bar{x}_r$ is not an integer. Then there exists an $1\le s\le n$ such that $\bar{x}_{rs}$ is a fraction. This implies that there is an $1\le r'\le m$ $r\ne r'$ such that $\bar{x}_{r's}$ is a fraction.

Assume that $\bar{x}_{r'}\le\bar{x}_r$. Consider the feasible solution $\mathbf{x}'$ obtained by setting $x'_{r's}=0$ and $x'_{rs}=\bar{x}_{rs}+\bar{x}_{r's}$.

Since the separable terms for $x_{r'}$ and x_r in the concave function g are identical and strictly concave, the assumption, $\bar{x}_{r'}\le\bar{x}_r$ implies that the objective function value of the new solution $\mathbf{x}'$ is less than the that of the optimal solution $\bar{\mathbf{x}}$. A contradiction.

Similarly, when $\bar{x}_r\le\bar{x}_{r'}$ we can derive a contradiction by showing that decreasing x_{rs} to zero and increasing $x_{r's}$ from $\bar{x}_{r's}$ to $\bar{x}_{r's}+\bar{x}_{rs}$ decreases the objective value. Hence $\bar{x}_i$ is an integer for every $i=1,\ldots,m$. □

Consider the following transportation problem

$$(P2.2)\qquad \min \sum_{\substack{i,j \text{ s.t.}\\ j \text{ nomem } S_i}} M x_{ij}$$
$$\sum_{j=1}^{n} x_{ij}=\bar{x}_i \quad i=1,\ldots,m$$
$$\sum_{i=1}^{m} x_{ij}=1 \quad j=1,\ldots,n$$
$$x_{ij}\ge 0 \quad i=1,\ldots,m,\ j=1,\ldots,n.$$

$(\bar{x}_{11},\bar{x}_{12},\ldots,\bar{x}_{mn})$ is a feasible solution of $(P2.2)$. Furthermore it is an optimal solution from Claim 1.

Let $(x^*_{11},x^*_{12},\ldots,x^*_{mn})$ be an integer optimal feasible solution of $(P2.2)$ which can be obtained in polynomial time by using, for example, the minimum cost flow algorithm of

[Orlin93]. x_{ij}^* is 0 or 1 since $\bar{x}_i$ are integers by Claim 2. $\mathbf{x}^*$ and $\bar{\mathbf{x}}$ have the same objective value for $(P2.1)$ since $x_i^* = \bar{x}_i$ for every $i = 1,\ldots,m$ and the value of the linear cost term is zero for both solutions.

Thus $(x_1^*, x_2^*, \ldots, x_m^*; x_{11}^*, x_{12}^*, \ldots, x_{mn}^*)$ is an optimal basic feasible solution of $(P2.1)$ and we may omit the integer constraints in $(P2.1)$.

From Claim 1 an integer optimal feasible solution of $(P2.1)$ is a feasible solution of **SET COVER.** It is now shown that an integer optimal solution of $(P2.1)$ is also an optimal solution of **SET COVER**. We prove that g is a strictly monotonic increasing function of the size of a cover. The size of a cover is equal to the number of *positive* (integer) elements in $(x_1, x_2, \ldots, x_m)$.

Lemma 2.2: Let p be an integer satisfying $0 < p < \min\{m, n\}$. Let $(x_1, x_2, \ldots, x_m)$ be a feasible solution of $(P2.1)$ with p positive elements and let $(x'_1, x'_2, \ldots, x'_m)$ be a feasible solution of $(P2.1)$ with $p+1$ positive elements. Then,

$$(x_1+1)^{1/n} + (x_2+1)^{1/n} + \cdots + (x_m+1)^{1/n} < (x'_1+1)^{1/n} + (x'_2+1)^{1/n} + \cdots + (x'_m+1)^{1/n}.$$

Proof: Consider the following two problems:

$$(P2.3) \qquad P = \max\ (y_1+1)^{1/n} + (y_2+1)^{1/n} + \cdots + (y_p+1)^{1/n}$$
$$y_1 + y_2 + \cdots + y_p = n$$
$$y_i \geq 1 \text{ integer } i = 1,\ldots,p$$

$$(P2.4) \qquad Q = \min\ (y_1+1)^{1/n} + (y_2+1)^{1/n} + \cdots + (y_{p+1}+1)^{1/n}$$
$$y_1 + y_2 + \cdots + y_{p+1} = n$$
$$y_i \geq 1 \text{ integer } i = 1,\ldots,p+1.$$

The vector consisting of the positive elements of $(x_1, x_2, \ldots, x_m)$ is feasible for $(P2.3)$. Hence,

$$(x_1+1)^{1/n} + (x_2+1)^{1/n} + \cdots + (x_m+1)^{1/n} \leq P.$$

Similarly,

$$(x'_1+1)^{1/n} + (x'_2+1)^{1/n} + \cdots + (x'_m+1)^{1/n} \geq Q.$$

So the lemma will follow if we prove that $P < Q$.

Since $(P2.3)$ is symmetric with respect to its variables and the objective is concave, the optimal solution to the relaxed problem has all variables equal to n/p. Therefore,

$$P \leq p\left(\frac{n}{p}+1\right)^{1/n}.$$

$(P2.4)$ is also symmetric but it involves minimization. Hence the optimal solution of the continuous version of $(P2.4)$ is achieved at any vertex, i.e., any solution vector with p 1's and a single $n-p$ as its entry. Thus,

$$Q \geq p\, 2^{1/n} + (n-p+1)^{1/n}.$$

To show $P < Q$, it suffices to prove

$$p\left(\frac{n}{p}+1\right)^{1/n} < p\, 2^{1/n} + (n-p+1)^{1/n},$$

or, equivalently,

$$\left(p\left(\frac{n}{p}+1\right)^{1/n}\right)^n < \left(p\, 2^{1/n} + (n-p+1)^{1/n}\right)^n. \qquad (2.2.1)$$

Using the equality

$$(p\,(\frac{n}{p})^{1/n})^n = p^n + n\,p^{n-1},$$

it follows that

$$(p\,2^{1/n} + (n-p+1)^{1/n})^n$$

$$= 2p^n + \binom{n}{n-1} p^{n-1} 2^{(n-1)/n} (n-p+1)^{1/n} + \cdots + \binom{n}{1} p\,2^{1/n} (n-p+1)^{(n-1)/n} + (n-p+1).$$

$$> p^n + \binom{n}{n-1} p^{n-1} 2^{(n-1)/n} (n-p+1)^{1/n} = p^n + n\,p^{n-1} 2^{(n-1)/n} (n-p+1)^{1/n}.$$

Since $n-p+1>1$,

$$p^n + n\,p^{n-1} 2^{(n-1)/n} (n-p+1)^{1/n} \ge p^n + n\,p^{n-1}.$$

Hence (2.2.1) follows and the proofs of Lemma 2.2 and hence of Theorem 2.1 are complete. □

The reduction from **SET COVER** implies that (PTP) is NP-hard even when g is separable and symmetric.

2.2 Two Polynomial Classes of (PTP)

2.2.1 (PTP) with a fixed number of sinks

By substituting $x_i = \sum_{j=1}^{n} x_{ij}$ in the function g, (PTP) can be written as follows:

$$\min g(x_{11}+x_{12}+\ldots+x_{1n}, \ldots, x_{m1}+x_{m2}+\ldots+x_{mn}) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\,x_{ij}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j=1,\ldots,n$$

$$x_{ij} \ge 0 \quad i=1,\ldots,m,\ j=1,\ldots,n.$$

Since $g(x_1,x_2,\ldots,x_m)$ is a concave function and $x_i = \sum_{j=1}^{n} x_{ij}$ is a linear function, g is a concave function of $\{x_{ij}\}$. Hence an optimal solution is attained at a vertex of the polyhedron of the feasible solution set. Such optimal solution is of the form,

$$x_{ij} = \begin{cases} b_j & \text{if } i = i_j \\ 0 & \text{if } i \ne i_j \end{cases}$$

for some i_j for $j=1,\ldots,n$.

There are $O(m^n)$ possible optimal solutions. Thus enumerating all of them is done in strongly polynomial time if the number of sinks n is fixed.

2.2.2 (PTP) with identical costs in rows or columns

Consider (PTP) with a transportation cost matrix in which the costs are identical in each row. That is, for each $i=1,2,\ldots,m$, there is a constant c_i such that $c_{ij}=c_i$ for all $j=1,2,\ldots,n$. The objective function is written as $g(x_1,x_2,\ldots,x_m) + \sum_{i=1}^{m} c_i\,x_i$, which is concave with respect to $(x_1,x_2,\ldots,x_n)$. An optimal solution is attained at a vertex of the polyhedron and is of the form,

$$x_i^* = \begin{cases} B\ (=b_1+b_2+\cdots+b_n) & \text{if } i = i' \\ 0 & \text{if } i \ne i' \end{cases}$$

for some $1 \le i' \le m$. An optimal solution is derived by evaluating the function values for $i' = 1, 2, \ldots, m$ and choosing an index i' giving the smallest value.

A similar observation applies when the costs are identical along each column. Namely, for each $j = 1, 2, \ldots, n$, there is a constant c_j such that $c_{ij} = c_j$ for all $i = 1, 2, \ldots, m$. In this case the linear cost term is the constant, $\sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} c_j b_j$ and hence the objective function is again concave in $(x_1, x_2, \ldots, x_n)$. The optimal solution is again of the form (*) and can be determined by comparing the function values for $i' = 1, 2, \ldots, m$ and choosing an index yielding the smallest value.

A special case the above polynomial class is when all c_{ij} are identical, $c_{ij} = c$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. In this case the linear term becomes a constant and can be deleted from the objective function. So without a linear term in the objective function the problem is easily solved. Thus the linear term in the objective function appears to be a factor making this problem hard.

3. A Strongly Polynomial Algorithm for (PTP) with Monge Cost Matrix and Fixed Number of Sources

In this section, we show that if the cost matrix (c_{ij}) has *Monge* property and m is fixed, (PTP) is solvable in strongly polynomial time.

We show first that (PTP) is solvable by enumerating the breakpoints of a multi-parametric transportation in Subsection 3.1. Subsection 3.2 includes a discussion and definition of *Monge* property. In Subsection 3.3, we show that the breakpoints of the multi-parametric transportation defined in Subsection 3.1, can be enumerated in strongly polynomial time if the cost matrix has Monge property and m is fixed.

The approach here could be viewed as a generalization of the approach in [TDG92] for $m = 2$, as matrices with only two rows (or two columns) always satisfy the Monge property (see e.g. [ACHS89]).

3.1 A Parametric Problem

Consider the following parametric transportation problem,

$$(TP(\alpha)) \qquad z(\alpha) = \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$
$$\sum_{j=1}^{n} x_{ij} = \alpha_i \quad i = 1, \ldots, m$$
$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, \ldots, n$$
$$x_{ij} \ge 0 \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where each α_i is nonnegative and $\alpha_1 + \cdots + \alpha_m = B$ (where, $B = b_1 + \cdots + b_m$).

It is known that on the domain $D = \{ \alpha : \alpha_i \ge 0 \text{ for } i = 1, 2, \ldots, m, \ \alpha_1 + \cdots + \alpha_m = B \}$, the optimal value of $(TP(\alpha))$, $z(\alpha)$, is a piecewise affine function (see e.g. [Murty83]). D is the union of disjoint polyhedral sub-domains, $D_1, D_2, \ldots, D_M$ so that $z(\alpha)$ is linear on each of the sub-domains.

Let $w(\alpha) = g(\alpha) + z(\alpha)$ (where g is the concave cost production function of (PTP)). Then the optimal value of (PTP) is equal to

$$\min_{\alpha \in D} w(\alpha) = \min_{k=1,\ldots,M} \min_{\alpha \in D_k} w(\alpha). \tag{3.1}$$

Note that D and its sub-domains D_k are (bounded) polytopes. Hence there is a finite set of points of D, $L=\{\alpha^1,\alpha^2,\ldots,\alpha^N\}$ such that each sub-domain D_k is the convex hull of a subset L_k of L. Each element of L is called a breakpoint of the parametric transportation problem, $(TP(\alpha))$.

Notice that $w(\alpha)$ is a concave function on each D_k since $z(\alpha)$ is affine on each D_k. So $\min_{\alpha\in D_k} w(\alpha)$ can be determined by evaluating function values of $w(\alpha)$ at the breakpoints $\alpha \in L_k$ and choosing a breakpoint giving the smallest value. From (3.1), the optimal objective function value of (PTP) is obtained by evaluating function values of $w(\alpha)$ at all the breakpoints of L and choosing a breakpoint giving the smallest value.

Thus, to solve (PTP), it suffices to find all breakpoints and the corresponding solution of $(TP(\alpha))$.

The parametric approach does not necessarily lead to an algorithm for the problem, unless it is known how to find all the breakpoints. Tuy et al [TDG92] showed that for $m=2$, $(TP(\alpha))$ has at most $n+1$ breakpoints that can be found in $O(n \log n)$ elementary operations. As discussed in Subsection 2.1, (PTP) has 2^n vertices when $m=2$. Thus the parametric approach represents a substantial improvement.

Although it is not explicitly observed in [TDG92], the reason for this efficiency is that every $2\times n$ cost matrix has *Monge* property [ACHS89] which is discussed in the following subsection.

Remark: In [TDG92], for $m=2$ the parametric transportation problem is derived in the context of *rank 2 condition*. It is also possible to derive $(TP(\alpha))$ using *rank m condition* which is a straightforward generalization of rank 2 condition. Using rank m condition, we can show that a more general class of concave minimization problem can be reduced into the problem enumerating the breakpoints of a parametric linear problem.

3.2. The Monge Property

Consider the transportation problem, (TP) defined with the cost matrix $C=(c_{ij})$:

$$(TP) \qquad \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\sum_{j=1}^{n} x_{ij}=a_i \quad i=1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij}=b_j \quad j=1,\ldots,n$$

$$x_{ij}\ge 0 \quad i=1,\ldots,m,\ j=1,\ldots,n.$$

C is said to have *Monge property* [Hof63] if there exists a permutation $((i_1,j_1),(i_2,j_2),\ldots,(i_{mn},j_{mn}))$ of indices of the cost matrix such that,

(M) for every $1\le i,k\le m$, $1\le j,l\le n$, whenever (i,j) precedes both (i,l) and (k,j), the corresponding entries in matrix C are such that $c_{ij}+c_{kl}\le c_{il}+c_{kj}$.

Any permutation satisfying (M) is called a *Monge sequence.* If C has a Monge property (and hence admits a Monge sequence), for any nonnegative integers, $a_1,\ldots, a_m, b_1,\ldots, b_n$ (with $\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$), (TP) is solved efficiently by a greedy algorithm. The following theorem is due to Hoffman [Hof63].

Theorem 3.1: A permutation $((i_1,j_1),(i_2,j_2),\ldots,(i_{mn},j_{mn}))$ is a Monge sequence of C if and only if for any nonnegative integers, $a_1,\ldots, a_m, b_1,\ldots, b_n$ with $\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$, the solution obtained by Algorithm Greedy (in Figure 1) is an optimal solution of TP.

```
Algorithm Greedy

begin
    u_i ← a_i for i=1,...,m; v_j ← b_j for j=1,...,n;
    for p = 1 to k do
    begin
        x_{i_k j_k} ← min {u_{i_k}, v_{j_k}};
        u_{i_k} ← u_{i_k} - x_{i_k j_k};
        v_{j_k} ← v_{j_k} - x_{i_k j_k};
    end; {for}
end;
```

Figure 1: Algorithm **Greedy**

The Monge property can be characterized in polynomial time. The algorithm of [ACH89] tests whether an $m \times n$ matrix has a Monge sequence in $O(m^2 n \log n)$.

Every $2 \times n$ matrix, C has a Monge sequence: renumber the columns of the matrix so that $c_{11} - c_{21} \le c_{12} - c_{22} \le \ldots \le c_{1n} - c_{2n}$. Then $((1,1),(1,2),\ldots,(1,n),(2,1),(2,2),\ldots,(2,n))$ is a Monge sequence as easily checked. The parametric transportation problem for $m = 2$ is,

$$(TP_2(\beta)) \qquad \min \sum_{i=1}^{2}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

$$\sum_{j=1}^{n} x_{1j} = \beta$$

$$\sum_{i=1}^{2} x_{ij} = b_j \quad j = 1,\ldots,n$$

$$x_{ij} \ge 0 \quad i = 1,2,\ j = 1,\ldots,n,$$

for $0 \le \beta \le B$ (only a single parameter is required by deleting a redundant constraint).

Theorem 3.2: ([TDG92]) (i) The breakpoints of the optimal value function in $(TP_2(\beta))$ are in the set, $\{\beta^0, \beta^2, \ldots, \beta^n\}$, where

$$\beta^0 = 0,\ \beta^{k+1} = \beta^k + b^k,\ k = 0, 1, \ldots, n-1.$$

(ii) If $\sum_{j=0}^{l-1} b_j \le \beta \le \sum_{j=0}^{l} b_j$ (with $b_0 = 0$) for some $0 \le l \le n-1$, then the following solution is optimal for $(TP_2(\beta))$:

$$x_{11} = b_1, x_{21} = 0; \ldots; x_{1l-1} = b_{l-1}, x_{2l-1} = 0;$$

$$x_{1l} = \beta - \sum_{j=0}^{l-1} b_j,\ x_{2l} = \sum_{j=0}^{l} b_j - \beta;$$

$$x_{1l+1} = 0, x_{2l+1} = b_{l+1}; \ldots; x_{1n} = 0, x_{2n} = b_n.$$

The algorithm presented in the following subsection reduces a parametric problem into a two-row problem. We then use the algorithm implied by Theorem 3.2 to solve the two-row problem.

3.3. An Algorithm

Consider Algorithm **PTP** presented in Figure 2. It consists of three main parts: initialization (line 1-3), breakpoint enumeration for $(TP(\alpha))$ (line 4) which is done by the procedure **Breakpoint-Finder** described in Figure 3, and optimization (line 5-6) which selects a breakpoint and a corresponding solution yielding the smallest objective value.

From the arguments of Subsection 3.1 the validity of the algorithm **PTP** immediately follows from the validity of the procedure **Breakpoint-Finder**.

Algorithm PTP

begin

1 set $I = \{1,2, \ldots, m\}$ and $J = \{1,2, \ldots, n\}$;
2 $Q \leftarrow \{ \alpha_1 \geq 0, \alpha_2 \geq 0, \ldots, \alpha_{m-1} \geq 0, \alpha_1+\alpha_2+ \ldots +\alpha_{m-1} \leq B\}$;
3 set $u_i = \alpha_i$ for $i=1,\ldots,m-1$, $u_m = B - (\alpha_1+\alpha_2+ \ldots +\alpha_{m-1})$ and $v_j=b_j$ for $j=1,\ldots,n$;

4 **call** Procedure **Breakpoint-Finder** (I,J);

5 evaluate the objective function value for the solution
corresponding to each breakpoint;
6 choose a solution of minimum value as an optimal solution of (*PTP*);

end;

Figure 2: Algorithm **PTP**

Procedure Breakpoint-Finder(I,J)

begin
if $|I| > 2$ **then do**
begin
1 select the cell (i,j) such that $i \in I$ and $j \in J$, which is
at the top of the Monge sequence;
{row deletion}
2 let $x_{ij} = u_i$; $Q \leftarrow Q \cup \{ u_i \leq v_j \}$;
3 $I \leftarrow I - \{i\}$; $v_j \leftarrow v_j - u_i$;
4 delete the cells in the row i from the Monge sequence;
5 **call** Procedure **Breakpoint-Finder** (I,J);
{column deletion}
6 Let $x_{ij} = v_j$; $Q \leftarrow Q \cup \{ v_j \leq u_i \}$;
7 $J \leftarrow J - \{j\}$; $u_i \leftarrow u_i - v_j$;
8 delete the cells in the column j from the Monge sequence;
9 **call** Procedure **Breakpoint-Finder** (I,J);
end; {if}
Else, then do
begin {solving two row problem}
10 say $I = \{1,2\}$ and $J = \{1, 2,\ldots, k\}$;
11 sort $\{ c_{11}-c_{21}, c_{12}-c_{22},\ldots,c_{1k}-c_{2k} \}$ in an increasing order and
let the sorted sequence be $\{ c_{1j_1}-c_{2j_1}, c_{1j_2}-c_{2j_2},\ldots,c_{1j_k}-c_{2j_k} \}$;
for p = 1 **to** k **do**
begin
12 set $x_{1j_1}= v_{1j_1}$, $x_{2j_1}=0$, ... , $x_{1j_{p-1}}=v_{1j_{p-1}}$, $x_{2j_{p-1}}=0$, $x_{1j_p}=u_1 - \Sigma_{q \leq p-1} v_{j_q}$,
$x_{2j_p}=\Sigma_{q \leq p} v_{j_q} - u_1$, $x_{1j_{p+1}}=0$, $x_{2j_{p+1}}=v_{j_{p+1}}$, ... , $x_{1j_k}=0$, $x_{2j_k}=v_{j_k}$;
13 $Q \leftarrow Q \cup \{ \Sigma_{q \leq p-1} v_{j_q} \leq u_1, u_1 \leq \Sigma_{q \leq p} v_{j_q}\}$, where v_{j_0} is defined to
be zero;
{calculation of breakpoints}
14 solve every subset of m-1 equations of Q;
15 accept a solution $(\alpha_1,\alpha_2, \ldots,\alpha_{m-1})$ as a breakpoints only if
it satisfies $\alpha_i \geq 0$ for all i and $\alpha_1+\alpha_2+ \ldots +\alpha_{m-1} \leq B$;
16 **return** the accepted breakpoints and the
corresponding solution $\{x_{ij}\}$;
end; {for}
end; {else}
end; {Procedure}

Figure 3: Procedure **Breakpoint-Finder**

The procedure **Breakpoint-Finder** applies the greedy algorithm to the parametric problem ($TP(\alpha)$) using the Monge sequence. At each iteration, the greedy algorithm needs to be combined with branching step to enumerate the possible outcomes of optimal solution as the right hand side are parametrized in terms of α.

At each branching step, the current problem branches into two sub-problems with smaller dimensions. This branching is recursively repeated until the current problem has only two rows. Then the two-row parametric problem is solved using Theorem 3.1.

Theorem 3.3: Procedure **Breakpoint-Finder** correctly returns all the breakpoints of ($TP(\alpha)$).

Proof: At each iteration the procedure considers the cell (i,j) of the current problem, which is at the head of the Monge sequence (line 1). Initially, the original parametric problem, ($TP(\alpha)$) is the current problem. By Theorem 3.1, the greedy optimal solution is such that,

$$x_{ij} = \begin{cases} u_i & \text{if } u_i \le v_j \\ v_j & \text{if } u_i \ge v_j . \end{cases} \tag{3.3.1}$$

Thus we need to consider two cases, $u_i \le v_j$ and $u_i \ge v_j$. When $u_i \le v_j$, we assign u_i to the cell (i,j) and the equation $u_i \le v_j$ is added to Q to specify the sub-domains on which the assignment is optimal (line 2).

Also if $u_i \le v_j$, all other cells than (i,j) of the row i will not be assigned any positive shipment in the greedy solution. Hence the row i is deleted from further consideration and the demand level of the column j needs to be decreased to $v_j - u_i$ (line 3). Accordingly all the cells of the row i are deleted from the Monge sequence (line 4).

Line 6 - 8 describe the similar procedure for the case, $u_i \ge v_j$.

Thus after the branching step, the current problem branches into two sub-problems; the one with reduced rows and the other with reduced columns. For each sub-problem, the branching is recursively repeated if the number of rows is >2 (line 5 and 9).

Note that u_i and v_j are the affine functions of α. Initially $u_i = \alpha_i$ and $v_j = b_j$. By induction it is easy to see that at each iteration, $u_i = (\rho_{i1}\alpha_1 + \rho_{i2}\alpha_2 + \cdots + \rho_{im}\alpha_m) - \rho_{i0}$ and $v_j = \sigma_{j0} - (\sigma_{j1}\alpha_1 + \sigma_{j2}\sigma_2 + \cdots + \sigma_{jm}\alpha_m)$, where ρ_{ik} and σ_{jl} are nonnegative integers for k, $l = 0,\ldots,m$.

So the linear equation, $u_i = v_j$ defines a hyperplane which divides the domain $D = \{\alpha : \alpha_i \ge 0 \text{ for } i = 1,2,\ldots,m-1,\ \alpha_1 + \cdots + \alpha_{m-1} \le B\}$ into two polyhedral sub-domains. Clearly on each sub-domain the greedy solution of x_{ij} defined in (3.3.1) is an affine function of α. Hence the cost contribution of the greedy solution of x_{ij} to the objective value $z(\alpha)$ is also an affine function of α on each sub-domain. Thus by induction on the number of iterations, it follows that the cost of the currently assigned shipment levels (x_{ij}'s with $(i,j) \notin I \times J$) is an affine function of $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ in the current sub-domain defined by the equations in Q.

So far we have shown that at each branching iteration, the procedure assigns the shipment level to an additional cell in optimal manner and generates equations of $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ so that the equations in Q describe a sub-domain on which the cost of the currently assigned shipment levels is an affine function.

This branching is repeated until we get a two-row problem (line 10). Then we can find the optimal parametric solution using Theorem 3.2. The solution (line 12) and the equations describing corresponding ranges added to Q (line 13) are obtained from Theorem 3.2 (ii) by simply substituting $n = k$, $\beta = u_1$, and $b_l = v_l$ for $l = 1,\ldots,k$. If there are k columns in the problem, there are k corresponding ranges. Clearly, in each of the ranges the cost incurred by the shipment levels assigned to the cells in the two-row problem is an affine function of the $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$.

After line 12, we obtain a complete solution $\{x_{ij} \mid i = 1,\ldots,m, j = 1,\ldots,n\}$ of ($TP(\alpha)$). The equations in Q after the execution of line 13 describe a corresponding sub-domain on which the objective function value $z(\alpha)$ is an affine function.

Note that the total number of equations in Q is at most $m+n$. Thus each sub-domain has at most $\binom{m+n}{m-1}$ vertices (breakpoints of $(TP\alpha)$)) which can be found by solving every subset of m equations of Q as in line 14. In line 15, we choose only the solutions which satisfies the valid conditions, namely, each α_i is nonnegative and $\alpha_1 + \cdots + \alpha_{m-1} \le B$.

Thus the procedure **Breakpoint-Finder** correctly generates the complete set of breakpoints of $(TP(\alpha))$ and the corresponding solutions. The theorem follows. □

From Subsection 3.1 and Theorem 3.3, we have the following corollary.

Corollary 3.4: Algorithm **PTP** is valid.

Now we show that Algorithm **PTP** is strongly polynomial.

Theorem 3.5: The complexity of **PTP** is $O(n^{2m-2})$ if we assume a function evaluation oracle providing a single evaluation as unit operation.

Proof: The initialization (line 1 - 3) of the algorithm can be done in $O(n)$ steps as can be easily checked. The number of unit operations required for the optimization (line 5-6) is linear in the number of breakpoints generated in the procedure **Breakpoint-Finder**: the number of function evaluations (line 5) is the same as the number of breakpoints and the minimum value of line 6 can be found in linear time in the number of function values by using the algorithm in [BFPRT72].

The number of breakpoints generated in the procedure **Breakpoint-Finder** is bounded by the number of elementary arithmetic operations executed by **Breakpoint-Finder**.

So to prove the theorem, it suffices to show that the running time of **Breakpoint-Finder** is $O(n^{2m-2})$.

Let $T(m,n)$ be the number of elementary operations taken by the procedure **Breakpoint-Finder** when a parametric problem has m rows and n columns. Then by the recursive structure of the procedure, we have,

$$T(m,n) \le T(m-1,n) + T(m,n-1). \qquad (3.3.2)$$

First we show that the number of elementary operations required to solve two-row problem is $T(2,n) = O(n^m)$. The sorting in line 11 can be done in $O(n\log n)$ steps. For each $p = 1,\dots,k$, line 12 to 13 can be done in $O(n)$ steps. The operations of line 14 and 15 can be done in $O(n^{m-1})$ steps since, as shown in the proof of Theorem 3.3, when m is fixed, the total number of the sets of equations to be solved is $O(n^{m-1})$. Each set of equations is solvable in $O(m^3)$ which is a constant as m is fixed.

Thus for each $p = 1,\dots,k$, it takes $O(n^{m-1})$ steps and hence $T(2,n) = O(n^m)$ as $k \le n$.

Now assume inductively that for every $k = 2,\dots,m-1$, $T(k,n) = O(n^{m+k-2})$. Then by (3.3.2) there is a constant A such that,

$$\begin{aligned} T(m,n) &\le T(m-1,n) + T(m,n-1) \\ &\le A\, n^{2m-3} + A\,(n-1)^{2m-2} \\ &= A\, n^{2m-2}(1/n + [(n-1)/n]^{2m-2}) \\ &\le A\, n^{2m-2}(1/n + (n-1)/n) \\ &= A\, n^{2m-2} \end{aligned}$$

So the theorem follows. □

Remark: A careful look at the algorithm and its complexity analysis indicates that we may assume a weaker form of oracle. The function evaluations are required solely to find a solution yielding the smallest objective value. However a smallest valued solution can be found without calculating function values explicitly if only we can determine the ordinal sizes of the function values of given solutions used in comparisons (see e.g. [BFPRT72]). Thus it suffices to assume an oracle to determine the relative size of the function values (rather than calculating the explicit function values) of given solutions.

4. Open Problems

An immediate open question is on the polynomiality of (PTP) when the number of sources is fixed with a cost matrix that does not necessarily have the Monge property. A related question is: how many breakpoints does the parametric transportation problem $(TP(\alpha))$ have when the number of sources is fixed? If the number of breakpoints is exponential in n, the approach using the parametric transportation problem can no longer provide an efficient method to solve (PTP) even when the number of sources is fixed.

It is also an interesting problem to improve the polynomial algorithm presented in this paper. For example, is it possible to find a $O(n)$ algorithms for the cases discussed in Subection 2.2 and Section 3?

A further research in somewhat different direction is to develop an efficient heuristic for the general case. No such method is known even if we make some additional assumptions on the concave production cost function such as symmetry and separability.

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