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Collapsing effects in numerical simulation of chaotic dynamical systems *

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Introduction. In control system design, whether the various subsystems are in discrete time or continuous time, the state space is usually regarded as a continuum. However, when the system is implemented, some subsystems may have a state space which is a subset of finite computer arithmetic. This is an important concern if a subsystem has chaotic behaviour, because it is theoretically possible for rich and varied motions in a *continuum* to collapse to trivial and degenerate behaviour in a finite and discrete state space [5]. This paper discusses new ways to describe these effects and reports on computer experiments which document and illustrate such collapsing behaviour.

Let f be a function $[0,1]$ onto $[0,1]$, $f(0)=0$ and $f(1)$ is equal to either 0 or 1. Such functions often arise as a source of chaotic behaviour in many one–dimensional or multi-dimensional systems [7],[8]. The classical examples include the logistic mapping $F(x)=4x(1-x)$, the mapping $G(x)=4x^3-3x^2$ or mappings from the family $H_l(x)=1-2^l|1/2-x|^l$. Denote by $\mathbf{L}_\nu$ the uniform $1/\nu$ lattice on $[0,1]$: $\mathbf{L}_\nu=\{0,1/\nu,2/\nu,\ldots,1\}$. For $x\in[0,1]$ and $k/\nu\le x<(k+1)/\nu$, for some $0\le k\le \nu-1$ denote the roundoff operator $[x]_\nu$ by

$$[x]_\nu=\begin{cases} k/\nu & \text{if} \quad k/\nu\le x\le \frac{k+1/2}{\nu},\\ (k+1)/\nu & \text{if} \quad (k+1/2)/\nu\le x\le \frac{k+1}{\nu}.\end{cases}$$

Let f_ν denote the mapping $\mathbf{L}_\nu\mapsto\mathbf{L}_\nu$ defined by $f_\nu(\xi)=[f(\xi)]_\nu$. This is a $\mathbf{L}_\nu$-*discretization* of f. A point $\xi\in\mathbf{L}_\nu$ is ν-*collapsing* if $f_\nu^n(\xi)=0$ or 1 for some finite n. Denote the set of ν-collapsing points by $Y(\nu;f)$. Let $p_\nu(f)$ denote the proportion of collapsing points from the lattice $\mathbf{L}_\nu$. That is $p_\nu(f)=\nu^{-1}\,\#(Y(\nu;f))$. This paper studies typical properties of sequences

$$\mathbf{P}(f)=p_1(f),p_2(f),\ldots,p_\nu(f),\ldots .$$

Let $\tilde p(\nu;F)$ be the proportion of collapsing elements from a random sample of 100 initial values $\xi\in\mathbf{L}_\nu$. The value of $\tilde p(\nu;F)$ is a reasonable statistical approximation to $p_\nu(F)$. Figures 1a and 2a respectively graph $p_\nu(G)$ and $2Yp_\nu(H_3)$ for the interval $2^{27}\le\nu\le 2^{27}+500$. Figures 3a, 4a graph $p_\nu(H_3)$ for other intervals of ν.

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Interesting features to note are that the dependence on ν is apparently highly irregular and not autocorrelated and that a high proportion of collapsing points occur at frequent ν. These features are an artifact of the interaction between the dynamical system induced by F and the discretization $[\cdot]_\nu : \mathbb{R} \mapsto \mathbf{L}_\nu$. It is desirable to find a simple model to explain this interaction both qualitatively and quantitatively.

The model. The main technical tools below are random mappings with a single attracting centre. Let us recall the definition. Let Δ be a positive value with $1 < \Delta < \kappa$. Define a random mapping $T_{\Delta,\kappa}$ by the formula

$$P(T_{\Delta,\kappa}(i) = j) = \begin{cases} \Delta/(\kappa+\Delta) & \text{if} \quad j = 0, \\ \dfrac{1}{\Delta+\kappa} & \text{otherwise.} \end{cases} \tag{1}$$

and the image of an element x_i is chosen independently of those of other elements of X.

Define the *collapsing component* Z of of this mapping as a random subset of $(0, 1, \ldots, \kappa)$ $Z_{\Delta,\kappa} = \{i \in E(\kappa) : T^n_{\Delta,\kappa} i = 0 \text{ for some } n\}$. Introduce the random variable

$$Q_{\Delta,\kappa} = \frac{\#(Z_{\Delta,\kappa})}{\kappa}. \tag{2}$$

That is, $Q_{\kappa,\Delta}$ is the proportion of elements of $(0, 1, \ldots, \kappa)$ belonging to the collapsing component of the mapping $T_{\Delta,\kappa}$.

Now we can formulate a slightly generalized form of the hypothesis suggested in [6]. Let f be a function $[0,1]$ onto $[0,1]$ which has a unique absolutely continuous measure μ_f. Furthermore, suppose that the density of this measure is positive and only the end-points of the interval $[0,1]$ are singular for the density, moreover, that for a given a certain $l \geq 2$ there exist limits $\alpha_0 = \lim_{\gamma\to 0} \mu([0,\gamma])\gamma^{1/l}$, and $\alpha_1 = \lim_{\gamma\to 1} \mu([1-\gamma, 1])\gamma^{1/l}$. These assumptions hold for functions $F(\cdot)$, $G(\cdot)$, $H_l(\cdot)$, $l \geq 2$ and many others, see [8] and references therein.

The hypothesis linking the proportion of collapsing points and the collapsing component of the random mapping $T_{\Delta,\kappa}$ is as follows:

Hypothesis 1 *There exist positive constants $a = a(f)$, $b = b(f)$ with the following property. For large N and $1 << n << N$, the statistical characteristics of the sequence $\mathbf{P}(N, n; f) = p_N(f), p_{N+1}(f), \ldots, p_{N+n}(f)$ are similar to those of the random variable $Q_{\Delta(\nu;f),\kappa(\nu;f)}$. The parameters $\Delta(\nu, f), \kappa(\nu, f)$ are determined by*

$$\Delta(\nu; f) = a(f)\frac{\sqrt{\nu}}{\ln \nu}, \qquad \kappa(\nu; f) = \left[b(f)\frac{\nu}{\ln(\nu)}\right],$$

if $l = 2$, and by

$$\Delta(\nu; f) = \frac{a(f)}{\nu^{1/l}}, \qquad \kappa(\nu; f) = \left[b(f)\frac{\nu}{\nu^{2/l}}\right],$$

if $l > 2$.

We have found no rigorous justification for this hypothesis. However the close agreement between its theoretical consequences and computational experiments strongly suggest that it does hold or, at least, with some close modification is valid.

Theoretical properties of the model. From results in [4] it follows easily that

Lemma 1 (a) *In the case $l = 2$ the sequence of random variables*

$$Q_{\lambda(\nu;f),\kappa(\nu;f)}\ln(\nu), \qquad \nu = 1,2,\ldots$$

converges in distribution to a random variable with the density $d_f(x) = Ad(Afx)$, where $d(x) = \dfrac{1}{\sqrt{2\pi}x^{3/2}}e^{-1/(2x)}, \quad 0 < x < \infty$, for some $A > 0$.

(b) *In the case $l > 2$ the sequence of the random variable*

$$Q_{\lambda(\nu;f),\kappa(\nu;f)}, \qquad \nu = 1,2,\ldots$$

converges in distribution to a random variable with the density

$$d_c^*(x) = \frac{c_f}{\sqrt{2\pi}}x^{-3/2}(1-x)^{-1/2}e^{-c_f^2(1-x)/(2x)}, \qquad 0 < x < 1$$

for some $c > 0$.

Experimental results. Let **S** be a finite set of non-negative values and define the *distribution function* of the set **S**

$$D(a, \mathbf{S}) = \frac{\#(\{s \in \mathbf{S} : s \le a\})}{\#(\mathbf{S})}, \qquad a \ge 0.$$

Hypothesis 1 and Lemma 1 imply that $D(a; \ln(N)\mathbf{P}(N, n; G))$ should be close, for reasonably large N and $1 << n << N$, to the probability distribution with density function $A(f)d(A(f)x)$, for some $A > 0$.

Figure 1b shows $D(a; \ln(N)\widetilde{\mathbf{P}}(N, n; G))$ for $N = 2^{27}$, $n = 500$ and the distribution function with density $d(x)$ (the smooth line). Here,

$$\widetilde{\mathbf{P}}(N, n; G) = \widetilde{p}(N; G), \widetilde{p}(N+1; G), \ldots, \widetilde{p}(N+n; G),$$

and the $\widetilde{P}_\nu$ were calculated as described in the introduction. In Figure 2b are shown $D(a; \widetilde{\mathbf{P}}(2^{27}, 500; H_3))$ and the distribution function with density $d_1^*(x)$) (smooth line). Another corollary of Hypothesis 1 is that various graphs of $D(a; \widetilde{\mathbf{P}}(N, 500; H_3))$ for different N should be all similar for sufficiently large N. In Figures 3b and 4b are shown $D(a; \widetilde{\mathbf{P}}(2^{13}, 500; H_3))$ and $D(a; \widetilde{\mathbf{P}}(2^{44}, 500; H_3))$.

Figure 1. $f(x) = 4x^3 - 3x \quad (-1 < x < 1)$ for the grid number $N : (2^{27}, 2^{27} + 500)$

(1a) (1b)

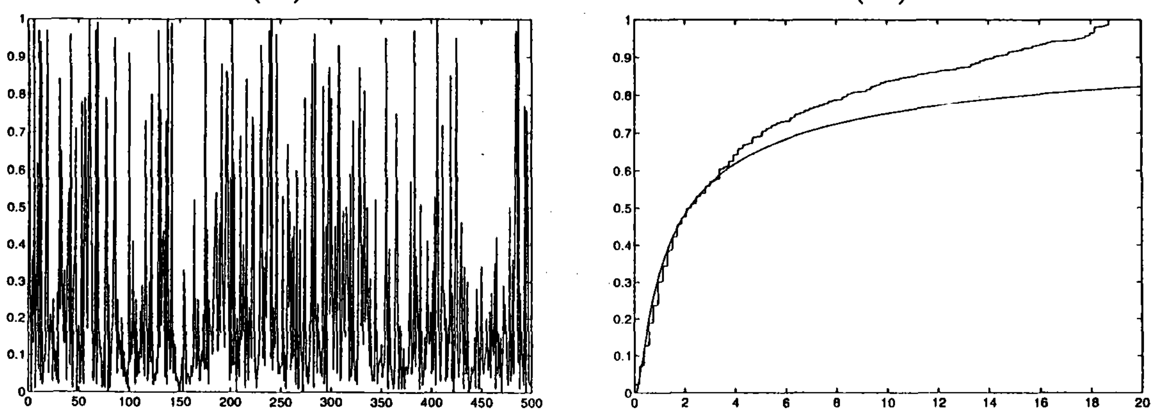


Figure 2. $f(x) = \begin{cases} 1 - 8\left(x - \frac{1}{2}\right)^3 & (\frac{1}{2} < x < 1) \\ 1 - 8\left(\frac{1}{2} - x\right)^3 & (0 < x \le \frac{1}{2}) \end{cases}$ for the grid number $N : (2^{27}, 2^{27} + 500)$

(2a) **(2b)**

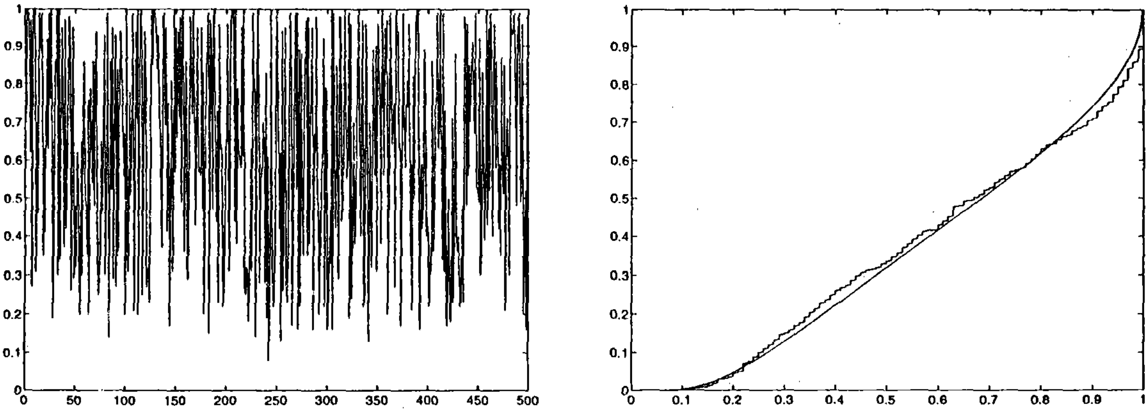


Figure 3. for the grid number $N : (2^{13}, 2^{13} + 500)$

(3a) **(3b)**

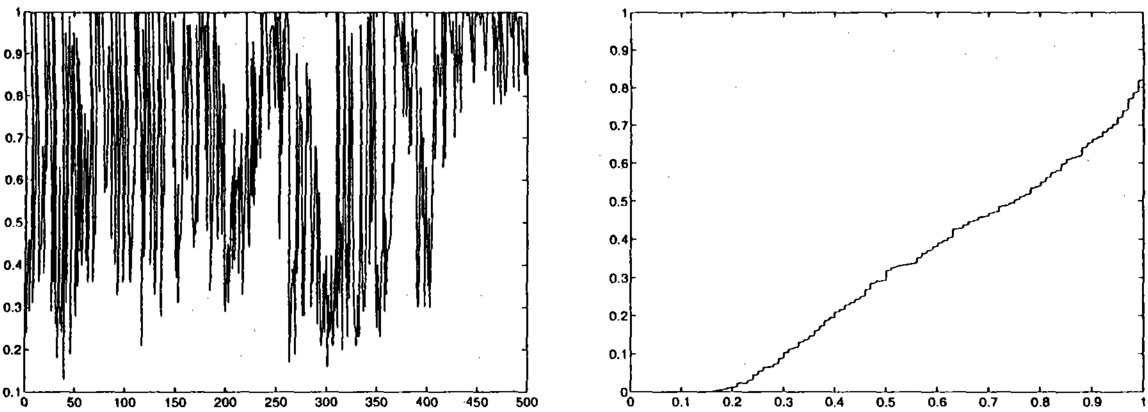
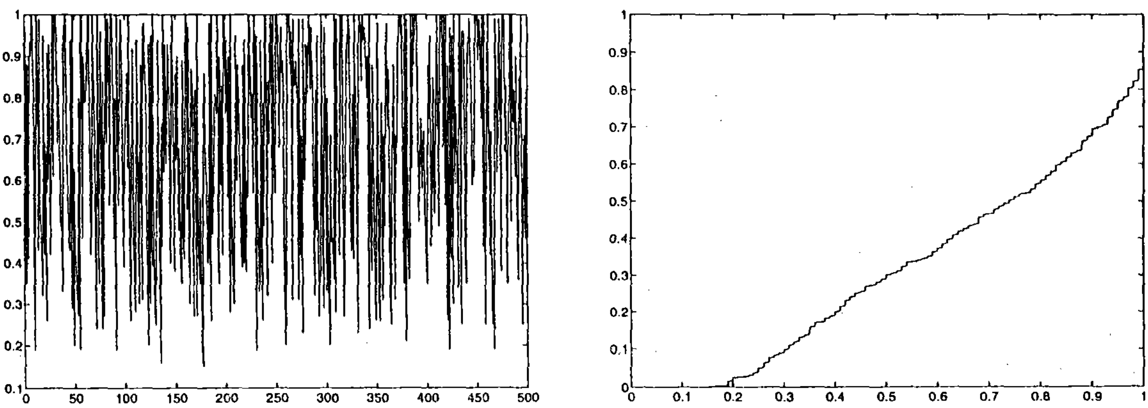


Figure 4. for the grid number $N : (2^{44}, 2^{44} + 500)$

(4a) **(4b)**



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