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Two-position Alignment of Strapdown Inertial Navigation System

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ABSTRACT

Some extended results in the study of two-position alignment for strapdown inertial navigation system are presented. In [1], an observability analysis for two-position alignment was done by analytic rank test of the stripped observability matrix and numerical calculation of the error covariance propagation using ten-state error model. In this paper, it is done by an analytic approach which utilizes the nonsingular condition of the determinant of simplified stripped observability matrix and by numerical calculation of the error covariance propagation accomplished in more cases than [1], and the twelve-state error model including vertical channel is used instead of ten-state error model. In addition, it is confirmed that this approach more clearly produces the same result as shown in the original work in terms of complete observability and there exist some better two-position configurations than [1] using the twelve-state error model.

I. INTRODUCTION

The initial alignment of strapdown inertial navigation system(SDINS) is an important process to determine the angular relationship between the navigation frame and the body frame before navigation starts. Stationary gyrocompassing is commonly used as a self-alignment method which utilizes its own instruments without the use of external equipment. Due to its importance, many researchers have studied the topic[1-6]. Recently, several papers which investigates system characteristics through an observability analysis were presented. Especially, multiposition alignment is proposed in [1] to improve the performance of stationary alignment of SDINS. The basic idea of the multiposition alignment is to improve the observability of SDINS system by changing the sensor position. Two-position alignment concentrated in this paper is performed by rotating the inertial measurement unit or equivalently by rotating vehicle body with respect to a single axis. The two-position alignment problem can be divided into two categories. One is the rotation with respect to which axis makes the system completely observable. The other is how much degrees of the relative angle between two positions minimizes the alignment errors for a given configuration. This problem was resolved in [1] by analytic rank test of the stripped observability matrix(SOM)[7] and numerical calculation of the error covariance propagation.

In this paper, to substitute rank test, we introduce the determinant of the SOM as a measure to determine the complete observability of piece-wise constant system and show that it produces an analytic solution. The advantage of this approach is that we can easily determine the singular case of SOM which means the system is not completely observable and this approach can present the relation of attitude angles in two-position alignment which make the system not completely observable.

In the next section we describe SDINS error model for the stationary alignment. In section III, we introduce an analytic approach to observability analysis using the determinant of simplified stripped observability matrix of piece-wise constant system. In section IV, we present the optimal two-position alignment using the numerical calculation of error covariance propagation. Finally, some conclusions will be given.

II. SDINS ERROR MODEL

An SDINS error model has a structural difference compared to the gimballed INS error model, that is, it contains the coordinate transformation matrix from the body frame to the navigation frame. For the purpose of stationary two-position alignment, we have extended the Lee, et al.'s SDINS error model[1] by adding the vertical channel error model. A local level NED(North-East-Down) frame is used as the navigation frame, and the alignment position is assumed to be precisely known so that the position error model may be omitted. The accelerometer and gyro errors are modeled as random biases. Then the SDINS stationary error model augmented with sensor errors can be represented by

$$\begin{bmatrix} \dot{x}_f(t) \\ \cdots \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} F & \vdots & T_i \\ \cdots & \cdots & \cdots \\ 0_{6\times6} & \vdots & 0_{6\times6} \end{bmatrix} \begin{bmatrix} x_f(t) \\ \cdots \\ x_a(t) \end{bmatrix} \equiv A_i x(t) \quad (1)$$

where $0_{6\times6}$ is zero matrix of indicated dimension and the state vectors, x_f and x_a consist of

$$x_f = [v_N, v_E, v_D, \psi_N, \psi_E, \psi_D]^T \quad (2a)$$

$$x_a = [\nabla_x, \nabla_y, \nabla_z, \varepsilon_x, \varepsilon_y, \varepsilon_z]^T \quad (2b)$$

where v and ψ are the velocity and attitude error, respectively; ∇ is the accelerometer error; ϵ is the gyro error; the subscripts x, y, and z denote the body axes. And system dynamic matrix F is represented by

$$F = \begin{bmatrix} 0 & 2\Omega_D & 0 & 0 & g & 0 \\ -2\Omega_D & 0 & 2\Omega_N & -g & 0 & 0 \\ 0 & -2\Omega_N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_D & 0 \\ 0 & 0 & 0 & -\Omega_D & 0 & \Omega_N \\ 0 & 0 & 0 & 0 & -\Omega_N & 0 \end{bmatrix} \equiv \begin{bmatrix} F_{11} & \vdots & F_{12} \\ \cdots & \cdots & \cdots \\ 0_{3\times3} & \vdots & F_{22} \end{bmatrix} \quad (3)$$

where g is the gravitational force; Ω represents the earth rate; the subscripts N, E, and D denote the north, east, and down components, respectively. The matrix T_i is defined as follows.

$$T_i = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{11} & C_{12} & C_{13} \\ 0 & 0 & 0 & C_{21} & C_{22} & C_{23} \\ 0 & 0 & 0 & C_{31} & C_{32} & C_{33} \end{bmatrix} \equiv \begin{bmatrix} C_b^n & \vdots & 0_{3\times3} \\ \cdots & \cdots & \cdots \\ 0_{3\times3} & \vdots & C_b^n \end{bmatrix} \quad (4)$$

where $C_b^n = \{C_{ij}\}_{i,j=1,2,3}$ is the coordinate transformation matrix from the body frame b to navigation frame n. Provided that the transformation matrix has roll(ϕ)-pitch(θ)-yaw(ψ) Euler convention, it can be represented by

$$C_b^n = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \quad (5)$$

where $c\psi = \cos\psi$, $c\theta = \cos\theta$, $c\phi = \cos\phi$, $s\psi = \sin\psi$, $s\theta = \sin\theta$, $s\phi = \sin\phi$.

The matrix T_i varies due to the change of the transformation matrix, thus SDINS system becomes the time varying system depending on the Euler angles. Since the constant attitude change can be provided arbitrarily, the time varying system can be considered as a piece-wise constant system, called a segment. The measured signals during the stationary alignment are the velocity errors. Hence the observation model can be written as follows.

$$z(t) = \begin{bmatrix} I_{3\times3} & O_{3\times3} & O_{3\times3} & O_{3\times3} \end{bmatrix} \begin{bmatrix} x_f(t) \\ \cdots \\ x_a(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix} = Hx(t) + n(t), n(t) \sim N(0, R) \quad (6)$$

where the measurement matrix H is always constant regardless of vehicle attitude change.

III. OBSERVABILITY ANALYSIS

Observability analysis of a dynamic system represents the efficiency of an estimator designed to estimate the states of the system. For the sake of observability analysis of a piece-wise constant system, we must check not only complete observability but also the degree of observability of the system. It is because the degree of observability of piece-wise constant system can vary according to the configuration of segments, and the complete observableness by the rank test may not provide enough information when implementing an estimator[8]. In this section, we prove the rank of SOM is equal to that of *Total Observability Matrix*(TOM), and derive the determinant of SOM in order to determine the complete observability. At first, consider a continuous piece-wise constant system as follows.

$$\begin{aligned} \dot{x}(t) &= A_i x(t), \quad i = 1, 2, \cdots, r \\ z_i(t) &= H x(t) \end{aligned} \tag{7}$$

where r is the number of segments. And define a measurement vector Z_i as follows.

$$Z_i(t) = V_i x(t), \quad i = 1, 2, \cdots, r \tag{8}$$

where $Z_i(t)$ consists of the vector $z_i(t)$ and its $n-1$ derivatives, and V_i is represented by

$$V_i = \left[H^T, (HA_i)^T, \cdots, (HA_i^{n-1})^T\right]^T. \tag{9}$$

Then we can construct a measurement vector $Z(t)$ of the piece-wise constant system as follows.

$$Z(t) = Vx(t) \tag{10}$$

where $Z(t)$ is the vector of all vectors $Z_i(t)$, $i = 1, 2, \cdots, r$, and V is the TOM represented by

$$V = \begin{bmatrix} V_1 \\ V_2 e^{A_1\Delta_1} \\ V_3 e^{A_2\Delta_2} e^{A_1\Delta_1} \\ \vdots \\ V_r e^{A_{r-1}\Delta_{r-1}} \cdots e^{A_1\Delta_1} \end{bmatrix} \tag{11}$$

where Δ_i is the time interval of segment i. And V_S is the SOM represented by

$$V_S = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_r \end{bmatrix} \tag{12}$$

Using the nonsingular condition of (11), we can determine the complete observability. However, it is difficult to calculate analytically because (11) contains the transition matrices for segments. Thus we suggest the following theorem related to a simpler transformation.

THEOREM 1. *The SDINS error model defined in* (1) *and* (6) *satisfies* $V_S = E\ V$
where E is an elementary row operation matrix.

PROOF : See the Appendix.

Complete observability of two-position alignment is determined by the nonsingular condition of the determinant of observability matrix, then theorem 1 tells that we can use the simpler form of the SOM V_S instead of the TOM V.

IV. OPTIMAL TWO-POSITION

When stationary fixed-position alignment is performed, the rank of the observability matrix obtained from the SDINS model (1) and (6) is 9, that is, the system is not completely observable. But the system could be completely observable by changing the coordinate transformation matrix without the use of additional measurements of the other states. The change of the transformation matrix can be provided by changing the attitude of the vehicle or equivalently by rotating the inertial measurement unit. After the alignment in an initial position, the successive alignment continued in the second position obtained by rotating SDINS with respect to one axis from the initial position once. Using the two-position alignment, it is possible to estimate all the state variables because the system becomes completely observable. Furthermore, since the SDINS error model used in fixed position alignment is not completely observable, the error variances of unobservable state variables remain constant or decrease very slowly. On the other hand, two-position alignment causes the system to be completely observable and results in a fast decrease of error variance as shown in [1]. In order to determine the complete observability, consider the SOM for two-position alignment as follows.

$$V_S = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{13}$$

By elementary row operation, a simplified SOM can be obtained from (13) as follows.

$$\tilde{V}_S = \begin{bmatrix} I_3 & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & F_{12} & C_b^n(1) & 0_{3\times3} \\ 0_{3\times3} & F_{22} & 0_{3\times3} & C_b^n(1) \\ 0_{3\times3} & F_{12} & C_b^n(2) & 0_{3\times3} \\ 0_{3\times3} & F_{22} & 0_{3\times3} & C_b^n(2) \end{bmatrix} \tag{14}$$

where I and 0 are identity and zero matrices of indicated dimensions, and $C_b^n(1)$ and $C_b^n(2)$ are the coordinate transformation matrices in the first and second position respectively. If we assume that $C_b^n(1)$ and $C_b^n(2)$ are orthogonal, then after a lot of manipulations we obtain the determinant value of $\tilde{V}_S^T \tilde{V}_S$ as follows.

$$
\begin{aligned}
J &= det(\tilde{V}_S^T \tilde{V}_S) \\
&= 8(2 - S_{22})^2[\Omega_N^2 g^4 (2 - S_{11}) + \Omega_N^2 \Omega_D^2 g^2 (2 - S_{11}) \\
&\quad -2\Omega_N^3 \Omega_D g^2 S_{13} + \Omega_N^4 g^2 (2 - S_{33})] \qquad (15) \\
&\quad +8(2 - S_{22})[-\Omega_N^2 g^4 S_{12}^2 - \Omega_N^2 \Omega_D^2 g^2 S_{12}^2 \\
&\quad +2\Omega_N^3 \Omega_D g^2 S_{12} S_{23} - \Omega_N^4 g^2 S_{23}^2]
\end{aligned}
$$

where

$$S_{ij} = C_b^n(1)(C_b^n(2))^T + C_b^n(2)(C_b^n(1))^T. \qquad (16)$$

Using (15), (16) and (5), we obtain the following theorem.

THEOREM 2. *Given the pair (A_i, H) of (1) and (6) where Ω_N is nonzero, let $J(\alpha_1, \alpha_2)$ be the positive value obtained by two-position in (15) and (16). Here α_1 and α_2 denote Euler angles for the first and second position, respectively. Then the nonsingular conditions of J for one axis rotation become,*

$$(1) \quad J(\psi_1, \psi_2) \neq 0 \qquad for \;\; \psi_2 \neq 2n\pi + \psi_1 \qquad (17a)$$

$$(2) \quad J(\theta_1, \theta_2) = 0 \qquad for\ all \;\; \theta_1,\ \theta_2 \qquad (17b)$$

$$(3) \quad J(\phi_1, \phi_2) \neq 0 \qquad for \;\; \phi_2 \neq 2n\pi + \phi_1 \qquad (17c)$$

where ϕ, θ and ψ denote roll, pitch and heading angles, respectively.

PROOF : See the Appendix.

In theorem 2, the nonsingular conditions of (17a) and (17c) are equal to full rank conditions of observability matrix, that is, it implies complete observability of the system. Thus it shows that the change of the heading or roll angle always results in a completely observable system, while the change of the pitch angle causes rank deficiency. This result shows that the two-position alignment using the twelve-state SDINS error model with the vertical channel error model improves the observability of the two-position alignment obtained by roll axis rotation, compared to the complete observability condition using the ten-state SDINS error model of [1].

Now, to evaluate the degree of observablilty of two position alignment according to the change of each rotation angle, error covariance of kalman filter is needed. For the system described in (1) and (6), the error covariance matrix P_i is obtained by calculating the discrete Riccati matrix equation as follows[1].

$$
\begin{aligned}
P_i^{-1}(k) &= (\Phi_i^T(k, k-1) P_i(k-1) \Phi_i(k, k-1) + Q)^{-1} \\
&\quad + H^T R^{-1} H \qquad k = 1, 2, \ldots, n \qquad (18)
\end{aligned}
$$

In this paper, $P_i(0)$, Q, and R are assumed as

$$
\begin{aligned}
P_1(0) &= diag\{(0.1ft/s)^2, (0.1ft/s)^2, (0.1ft/s)^2, (1°)^2, \\
&\quad (1°)^2, (1°)^2, (100\mu g)^2, (100\mu g)^2, \\
&\quad (100\mu g)^2, (0.02°/h)^2, \\
&\quad (0.02°/h)^2, (0.02°/h)^2\} \\
Q &= diag\{(5\mu g)^2, (5\mu g)^2, (5\mu g)^2, (0.01°/h)^2, \\
&\quad (0.01°/h)^2, (0.01°/h)^2, 0, 0, 0, 0, 0, 0\} \\
R &= diag\{(0.01ft/s)^2, (0.01ft/s)^2, (0.01ft/s)^2\}.
\end{aligned}
$$

And the number of iteration performed for calcurating P_i is 600 which is equivalent to 600 seconds in time-scale. At 300 seconds, rotation happens.

Fig. 1. a and fig. 1. b represent 1σ values of ψ_E and ψ_D according to initial pitch angle change when the two-position alignment is accomplished by rotating 180 degrees with respect to roll axis. They show that when initial pitch angle is not 0 degree, the twelve-state error model gives better degree of observability of pitch angle error compared with the ten-state error model, and when initial pitch angle is in small range, the degree of observability of heading angle does not vary much about two models. Fig. 2. a and fig. 2. b compare roll axis rotation with heading axis rotation. They verify that the roll axis rotation excluding the case that initial pitch angle is near 0 degree slightly improves the degree of observability of the two-position alignment than the heading axis rotation.

Fig. 3. a and fig. 3. b show the 1σ values of ψ_D in case of two position alignment obtained by the heading axis rotation according to the change of the initial roll and pitch angle respectively. They show that the optimal rotation angle with respect to the heading axis is 180 degrees regardless of the initial roll or pitch angle change. Fig. 4. a and fig. 4. b represent the 1σ values of ψ_D according to the rotation angle with respect to the roll axis and the initial pitch and heading angle. fig. 4. a shows that in case that the initial heading angle is 0 degree and the initial pitch angle is not 0 degree, the optimal rotation angle with respect to the roll axis is 180 degrees. Fig. 5. a and fig. 5. b show the 1σ values of ψ_D according to the rotation angle with respect to the pitch axis and the initial heading angle respectively. Fig. 5. a verifies that the change of only pitch angle has no effect on error estimation. Also, fig. 4. b and fig. 5. b show that the effect of roll and pitch axis rotation may be reversed according to the initial heading angle.

Consequently, we can conclude that the optimal two-position which satisfies complete observability and minimizes alignment errors can be obtained by rotating SDINS by 180 degrees with respect to the roll axis when initial pitch angle in SDINS is in small range but zero degree. But from fig. 3. a and fig. 4. a, the rotation with respect to roll axis is more sensitive near 180 degrees than the rotation with respect to heading axis. Therefore, we can see the rotation with respect to heading axis is more stable and easier to implement than the rotation with respect to roll axis.

V. SUMMARY AND CONCLUSIONS

An optimal two-position alignment for improving the performance of stationary alignment of SDINS has been investigated by an analytic and numerical approach. Its performance has been examined by analytic observability analysis using the nonsingular condition of the determinant of the simplified stripped observability matrix and numerical calculation of error covariance propagation. From the nonsingular condition, it is shown that the two-position alignment obtained by rotating SDINS with respect to the roll and heading axis can always make the system completely observable, while the rotation with respect to the pitch axis result in a rank deficiency of observability matrix. And from the numerical calculation of error covariance propagation, it is shown that the twelve-state error model produces better performance than the ten-state error model, and the rotation of 180 degrees with respect to the roll axis produces better performance than the rotation with respect to the heading axis in case that there exists small pitch angle in initial condition. But if initial pitch angle is zero degree, the rotation with respect to heading axis is more stable and easier to implement than roll axis.

Above results show that the observability analysis using the nonsingular condition of the determinant of the simplified stripped observability matrix is very efficient and easy to determine the complete observability of continuous piece-wise constant system, and the twelve-state error model presents optimal rotation angle in more cases then the ten-state error model. And they can be applied to design a piece-wise constant system such as SDINS.

APPENDIX

To prove theorem 1, two lemmas are introduced and proved.

LEMMA 1. *For arbitrary integer k,l and the system dynamics matrix A defined in (1) satisfies*

$$A_i^k A_j^l = A_j^{k+l} \qquad for\ \forall 1 \le i,j \le r-1. \tag{A1}$$

PROOF OF LEMMA 1 : Define M^k such as

$$M^k \equiv \begin{bmatrix} F^k & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \tag{A2}$$

which satisfies

$$M^k A_i = A_i^{k+1},\ and\ A_i M^k = M^{k+1}. \tag{A3}$$

Then, the followings are satisfied.

$$\begin{aligned} A_i^k A_j^l &= M^{k-1} A_i M^{l-1} A_j \\ &= M^{k-1} \begin{bmatrix} F & \vdots & T_i \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} F^{l-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} A_j \\ &= M^{k-1} M^l A_j \\ &= M^{k+l-1} A_j = A_j^{k+l} \end{aligned} \tag{A4}$$

LEMMA 2.

$$\prod_j \left[I + \sum_{k=1}^{\infty} A_j^k \frac{\Delta_j^k}{k!} \right] \qquad for\ \forall\ 1 \le j \le r-1 \tag{A5}$$

can be represented by I and the linear combination of finite series of $A_1, \cdots, A_{r-1}$.

PROOF OF LEMMA 2 : for arbitrary j,

$$\begin{aligned} &\left[I + \sum_{k=1}^{\infty} A_j^k \frac{\Delta^k}{k!} \right] \left[I + \sum_{k=1}^{\infty} A_{j-1}^k \frac{\Delta^k}{k!} \right] \\ &= I + \sum_{k=1}^{\infty} A_j^k \frac{\Delta_j^k}{k!} + \sum_{k=1}^{\infty} A_{j-1}^k \frac{\Delta_j^k}{k!} \\ &+ \sum_{k=1}^{\infty} A_j^k \frac{\Delta_j^k}{k!} \sum_{k=1}^{\infty} A_{j-1}^k \frac{\Delta_j^k}{k!}. \end{aligned} \tag{A6}$$

By the result of Lemma 1,

$$\sum_{k=1}^{\infty} A_j^k \frac{\Delta_j^k}{k!} \sum_{k=1}^{\infty} A_{j-1}^k \frac{\Delta_j^k}{k!} = \sum_{k=2}^{\infty} A_{j-1}^k f(k) \tag{A7}$$

where $f(k)$ denotes a constant coefficient.

Substitute (A9) into (A8), (A8) becomes

$$I + \sum_{k=1}^{\infty} A_j^k \frac{\Delta_j^k}{k!} + \sum_{k=1}^{\infty} A_{j-1}^k g(k) \tag{A8}$$

where $g(k)$ denotes a constant coefficient.
Hence, (A7) can be represented as

$$I + \sum_{k=1}^{\infty} A_1^k a_{1k} + \cdots + \sum_{k=1}^{\infty} A_j^k a_{jk} \tag{A9}$$

where $a_{ik}(i = 1, \cdots, j)$ denote a constant coefficient. And by virtue of the Caley-Hamilton theorem, (A9) can be represented as

$$I + \sum_{k=1}^{r-1} A_1^k a'_{1k} + \cdots + \sum_{k=1}^{r-1} A_j^k a'_{jk}. \tag{A10}$$

where $a'_{ik}(i = 1, \cdots, j)$ denote a constant coefficient. Hense, lemma 2 can be proved.

PROOF OF THEOREM 1 : In the TOM of (11), the transition matrix of segment i can be represented by

$$e^{A_i \Delta_i} = I + \sum_{k=1}^{\infty} A_i^k \frac{\Delta_i^k}{k!} \tag{A11}$$

where I and 0 are twelfth order identity and sixth order zero matrices respectively; A_i is the constant matrix; Δ_i is the time interval of segment i; F and T_i is given by (1). we note that T_i varies due to vehicle attitude change but F is invariant regardless of the segment change. Inserting (9) and (A11) into (11) yields as follows.

$$V = \begin{bmatrix} \begin{bmatrix} H \\ HA_1 \\ HA_1^2 \\ \vdots \\ HA_1^{n-1} \end{bmatrix} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix} H \\ HA_2 \\ HA_2^2 \\ \vdots \\ HA_2^{n-1} \end{bmatrix} \left[I + \sum_{k=1}^{\infty} A_1^k \frac{\Delta_1^k}{k!} \right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{bmatrix} H \\ HA_r \\ HA_r^2 \\ \vdots \\ HA_r^{n-1} \end{bmatrix} \left[I + \sum_{k=1}^{\infty} A_{r-1}^k \frac{\Delta_j^k}{k!} \right] \cdots \left[I + \sum_{k=1}^{\infty} A_1^k \frac{\Delta_1^k}{k!} \right] \end{bmatrix} \tag{A12}$$

Lemma 1 and lemma 2 show that we can find nonsingular matrix E such that $V_S = E\ V$ where E denotes elementary row operation matrix. This completes the proof.

PROOF OF THEOREM 2 : In (15), we obtain $(S_{22} - 2)$ term as follows.

$$\begin{aligned} S_{22} - 2 = & \ 2(c\theta_1 c\theta_2 s\psi_1 s\psi_2 - 1) \\ & +2(c\phi_1 s\theta_1 s\psi_1 - c\psi_1 s\phi_1)(c\phi_2 s\theta_2 s\psi_2 - c\psi_2 s\phi_2) \\ & +2(s\phi_1 s\theta_1 s\psi_1 + c\psi_1 c\phi_1)(s\phi_2 s\theta_2 s\psi_2 + c\psi_2 c\phi_2) \end{aligned} \tag{A13}$$

In case of the two-position alignment obtained by the heading axis rotation, that is, when $\phi_1 = \phi_2 = 0$, and $\theta_1 = \theta_2 = 0$ are satisfied, we obtain the simplified equation of (A13) as follows.

$$\begin{aligned} S_{22} - 2 &= 2(s\psi_1 s\psi_2 + c\psi_1 c\psi_2 - 1) \\ &= 2(c(\psi_1 - \psi_2) - 1) \end{aligned} \tag{A14}$$

where we can obtain the nonsingular condition of J according to the heading angle change. The similar way can be applied to the other axis rotations.

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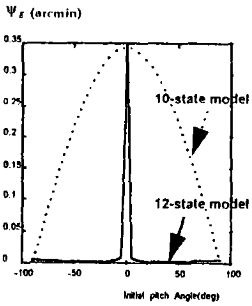

fig. 1. a 1σ values of ψ_E according to initial pitch angle in case that 2nd roll angle is 180 degree

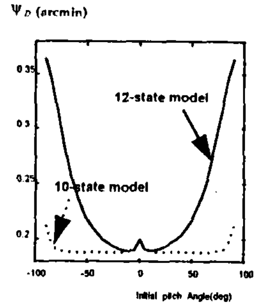

fig. 1. b 1σ values of ψ_D according to initial pitch angle in case that 2nd roll angle is 180 degree

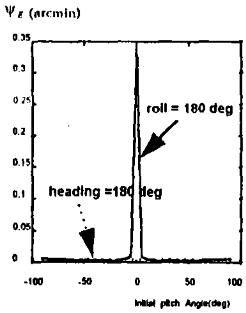

fig. 2. a 1σ values of ψ_E of 12-state SDINS error model according to initial pitch angle

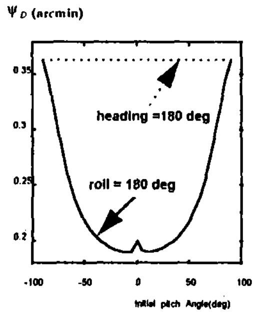

fig. 2. b 1σ values of ψ_D of 12-state SDINS error model according to initial pitch angle

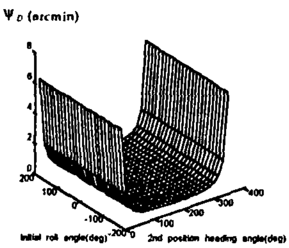

Fig. 3.a 1σ values of ψ_E according to heading axis rotation and initial roll angle ($\phi_1 = \phi_2, \theta_1 = \theta_2 = 0, \psi_1 = 0$)

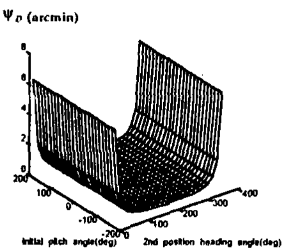

Fig. 3.b 1σ values of ψ_D according to heading axis rotation and initial pitch angle ($\phi_1 = \phi_2 = 0, \theta_1 = \theta_2, \psi_1 = 0$)

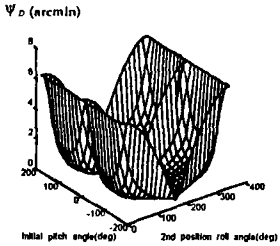

Fig. 4.a 1σ values of ψ_E according to roll axis rotation and initial pitch angle ($\phi_1 = 0, \theta_1 = \theta_2, \psi_1 = \psi_2 = 0$)

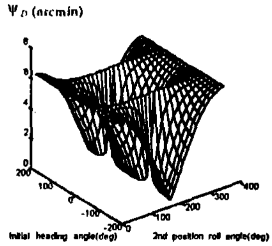

Fig. 4.b 1σ values of ψ_D according to roll axis rotation and initial heading angle ($\phi_1 = 0, \theta_1 = \theta_2 = 0, \psi_1 = \psi_2$)

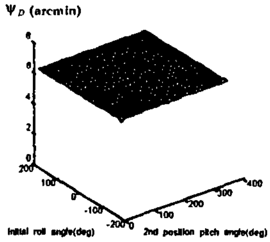

Fig. 5.a 1σ values of ψ_E according to pitch axis rotation and initial roll angle ($\phi_1 = \phi_2, \theta_1 = 0, \psi_1 = \psi_2 = 0$)

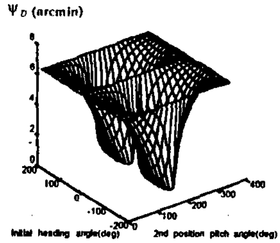

Fig. 5.b 1σ values of ψ_D according to pitch axis rotation and initial heading angle ($\phi_1 = \phi_2 = 0, \theta_1 = 0, \psi_1 = \psi_2$)