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A New Approach to Robustness Bounds Using Lyapunov Stability Concept

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Abstract

In this paper, the new approach and technique are introduced and derived from the original Lyapunov direct method which is used to decide the stability of stystem conveniently. This proposed technique modifies the formal concepts of the sufficient conditions of Lyapunov stability and is able to generate the methods for the robust design of control systems.

Also, it applies to the dynamic systems with bounded perturbations and the results of the computer program using the new concept are compared with those of previous researh papers and conventional Lyapunov direct method. It is possible to recognize the practical improvements of the estimation of robustness bounds of the systems.

1. Description of the New Approach Method to Robustness

The objective of the current investigation is the development of a new Lyapunov-based technique for the robust design of control systems subject to structured perturbations. The proposed technique provides better results for two basic robust control design problem, the robust stability problem and the uncertain system stabilization problem. This new approach involves application of the Lyapunov direct method to control design for time-variant, nonlinear systems with bounded perturbations. Lyapunov direct methods are one of the most popular way to determine the stability of the systems. This new approach to the determination of robustness bounds is now introduced, accompanied by consideration of improved stability criterion for relaxing Lyapunov stability conditions. The Lyapunov-based approach technique is summarized and proposed directions for further researches.

2. Introduction of Relaxing Lyapunov Stability Condition

In this paper, new criterion of determining the system stability originated from Lyapunov direct method is introduced and called relaxing Lyapunov stability condition. The original Lyapunov direct method is convenient way to determine the system stability without solving the system equation even if the systems are time-variant or nonlinear cases. A new throrem is proved and described in this paper concisely as follows:

<u>Theorem</u> Consider a system given

$$\frac{dX}{dt} = F(X, t) \qquad where \quad F \in C^{(0,1)}_{(t,x)} (I_t^+ \times R^n)$$

If there exist

a) a continuously differentiable and locally positive-definite function V(t,X) and

b) a bounded function T(t,X) defined for $t \in [0, \infty]$, $X \in B_r$, and having a positive lower bound such that the function

$V^{*}(t,X) = \int_{t-T(t,X)}^{t} V(\tau, X(\tau, t, X))d\tau$ exists, the condition $V^{*}(t,X) \leq 0$ is fulfilled.

Proof) The proof of this theorem differs from the classical proof of the Lyapunov theorem. It is the sufficient condition to prove stability for a selected initial instants t_0, and $t_0=0$ is selected in this case. Because of the definition of $V(t,X)$, there exists a continuous and positive-definite function $W(X)$ such that

$V(t,X) \geq W(X) > 0$, for $X \neq 0$, $X \in B_r$

$V(t,0)=W(0)=0$.

Choose the bounded range $S_\varepsilon = \{ \|X\| = \varepsilon \}$ such that $S \in B_r$. The lower bound of $W(x)$ on S_ε is obtained at a certain point X^* of S_ε, that is,

$$\inf_{X \in S_\varepsilon} W(X) = W(X^*) = \alpha > 0.$$

As $V(0,X)$ is continuous and $V(0,0)=0$, it is concluded that there exists a neighborhood $\|X\| < \delta_2 < \varepsilon$, such that for every $\|X_0\| < \delta_2$, $0 \leq V(0,X_0) < \alpha/2$. And T denotes the upper bound of $T(t,X)$:

$$\sup_{[0,\infty) \times B_r} T(t,X) = T .$$

Now δ_3 is arbitrarily chosen so that for every X_0 satisfying $\|X_0\| < \delta_3$,

$$\left| \int_0^u V(\tau, X(\tau, 0, X_0))d\tau \right| < \frac{\alpha}{2} \quad \text{for} \quad u \in [0,T] .$$

Denoting that $\delta_4 = \min(\delta_2, \delta_3)$, for $\|X_0\| < \delta_4$, $0 \leq V(0,X_0) < \alpha/2$ and

$$\left| \int_0^u V(\tau, X(\tau, 0, X_0))d\tau \right| < \frac{\alpha}{2} \quad \text{for} \quad u \in [0,T] .$$

Thus consider an arbitrary non-trivial solution with initial condition X_0, such that $\|X_0\| < \delta_4$, given that trajectory of this solution remains entirely inside of the range S_ε, that is,

$\|X(t,0,X_0)\| < \varepsilon$, for $t \in [0,\infty)$.

Then, make an assumption, that is, at some instant $t=t^*$, the point of trajectory is for the first time located on S_ε:

$\|X(t,0,X_0)\| < \varepsilon$ for $t \in [0,t^*)$

$\|X(t^*,0,X_0)\| = \varepsilon$.

The value of $V(t,x)$ is obtained by caculating

$$V(t^*, X(t^*,0,X_0)) = V(0,X_0) + \int_0^{t^*} V(\tau, X(\tau,0,x_0))d\tau$$

The integral on the right-hand side can be transformed to summation, writing

$$\int_0^{t^*} V(\tau, X(\tau,0,X_0) = \sum_{i=0}^{m-1} \int_{\tau_{i+1}}^{\tau_i} V(\tau, X(\tau,\tau_i,X_i))\,d\tau$$

with

$X(t^*,0,X_0)=X^*$, $t^*=\tau_0$, $\tau_{i+1} < \tau_i$, $X_i \in B_r$.

Using the definition in Theorem, all of the above integration

$$V(t^*, X(t^*,0,X_0)) \leq V(0,X_0) + \left| \int_0^{\tau_{m-1}} V(\tau, X(\tau,0,X_0))d\tau \right|$$

$$V(t^*, X(t^* 0, X_0)) < \alpha \quad .$$

In definition, $X(t^*,0,X_0) \in S_\varepsilon$ and $V(t^*, X(t^*,0,X_0)) \geq \alpha$, it contradicts and proves the theorem, that is, $X(t^*,0,X_0)$ is always remaining inside the boundary and the system is asymptotically stable judging by the Lyapunov stability concept.

3. Investigation of the Numerical Approach

In the actual dynamic systems, some numerica approaches demonstrate the application of the relaxing Lyapunov conditions. And it shows the improvement of robustness bounds. One example is drawn from other article and it is easy to compare the result. Systems with structured perturbations are introduced for the purposes of problem analysis with the proposed numerical and computational techniques. Fortran is the principal computer language used for programming, in conjunction with a symbolic algebraic calculation directed at the analytical solution of complicated mathematical problems. This application serves to minimize the large truncated errors inthe computer calculation, while providing methods for achieving exact solutions of problems. The numerical procedures can be simply described as follows:

A) First, the quadratic form is selected as the

Lyapunov candidate function. The bounds of uncertainty are defined by calculating the Lyapunov derivatives.

B) Second, the new bounds of uncertainty are selected, with the function $V^*(X)$ considered for the initial conditions belonging to a unit sphere.

C) For these initial conditions, for which the Lyapunov derivative remains negative, there is no need to find out a solution. For the remaining solutions, integration is extended until the function derivative is negative.

The following two examples are analyzed to verify the effectivity of the proposed numerical procedures.

Example 1. Consider the system

$$\frac{dX}{dt} = A(X) + g(t,X)GX , \quad X \in R^2$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} ,$$

the bound k on g(t,X) is such that if $|g(t,X)| < k$, the trivial solution of system equation is asymptotically stable. A quadratic form of the Lyapunov function V is chosen as Lyapunov candidate function $V=X^TPX$, $dV/dt=X^T(A^TP+PA)X$. Solving for the matrix P to obtain the best Lyapunov function results in $V = X_1^2 + X_1X_2 + X_2^2$.

Then, V is differentiated with respect to time,

$$\frac{dV}{dt} = 2X_1\dot{X}_1 + \dot{X}_1X_2 + X_1\dot{X}_2 + 2X_2\dot{X}_2 .$$

and replace the time derivatives of state variable :

$$\begin{aligned}\frac{dV}{dt} &= 2X_1X_2 + X_2^2 + X_1[(-1+g)X_1 - X_2] \\ &\quad + 2X_2[(-1+g)X_1 - X_2] \\ &= -[(1-g)X_1^2 + (1-2g)X_1X_2 + X_2^2] < 0\end{aligned}$$

If dV/dt fulfills the condition $dV/dt<0$, then the above equation must fulfil the following conditions:

a) $1 - g > 0$, $g < 1$, and

b) $(1-2g)^2 - 4(1-g) < 0$, $|g| < \frac{\sqrt{3}}{2} = 0.8660254$.

If the case is positive, then the system is asymptotic stable. The numerical procedure introduced previously is based on the result $|g| =$ 0.96, signifying a 10% improvement in the estimate of robustness bound.

Example 2) Now, the three-dimensional closed-loop system studied in Siljak, D.D. paper [12] previously is:

$$\overline{S} : \dot{X} = \begin{bmatrix} -2+k_1 & 0 & -1+k_1 \\ 0 & -3+k_2 & 0 \\ -1+k_1 & -1+k_2 & -4+k_1 \end{bmatrix} X .$$

This case is a two-degree of freedom dynamic system with two structured perturbations, k_1 and k_2. Applying the Lyapunov direct method, $V= X^TPX$ and $dV/dt=X^T(A^TP+PA)X$ and the equation $A^TP+PA=-I$ is solved, using the symbolic calculation package software. It follows that

$$P = \begin{bmatrix} \frac{2}{7} & \frac{9}{476} & -\frac{1}{14} \\ \frac{9}{476} & \frac{83}{476} & -\frac{11}{476} \\ -\frac{1}{14} & -\frac{11}{476} & \frac{1}{7} \end{bmatrix}$$

, where P is the exact solution and positive-definite. Then, dV/dt is calculated by the matrix

$$A^TP + PA = \begin{bmatrix} \frac{3k_1-7}{7} & -\frac{25k_2+2k_1}{476} & \frac{2k_1}{7} \\ -\frac{25k_2+2k_1}{476} & \frac{36k_2-119}{119} & \frac{57k_2-2k_1}{476} \\ \frac{2k_1}{7} & \frac{57k_2-2k_1}{476} & \frac{k_1-7}{7} \end{bmatrix}$$

If the matrix A^TP+PA had been negative-definite, the selected system would always be asymptotically stable. For A^TP+PA to be negative-definite, the regions k_1 and k_2 should be $|k_1| < 1.60$ and $|k_2| < 2.74$. For the current investigation, the regions considered above are drafted to be rectangular spaces. Using the same proposed technique, the k_1 and k_2 is extended. For $|k_1| < 1.60$, the region of $|k_2|$ is extended to 2.97. As may be seen from the results, the $|k_2|$ region is extended with an 8.3% improvement in robustness estimate.

5. Conclusions

The new proposed technique estimate the

stability region of the dynamic systems with bounded perturbations have been presented in this paper. New introducing stability criteria and conditions were presented and considered for chosen examples. The theorems were formulated to serve as a basis for both analytical and numerical procedures. The results demonstrateed improvements of the bounds in some extent and asymptotic stability for selected ranges of parameters. This new technique could be extended to multi-dimensional and multi-degree of freedom cases incorporated into standard design procedures based on the Lyapunov stability concept. Next, the future working area will be the developments of the selected actual and practical models to improve the efficient numerical procedures and robustness bounds.

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