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Derivation of a Group of Lyapunov functions associated with the System Energy

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Abstract

Most of the theorems of nonlinear stability is based on the Lyapunov stability theory. The Lyapunov function method is most well-known and provides precise and rigorous theoretical backgrounds. However, the conventional approach to direct stability analysis has been performed without taking account of damping effects. For accurate stability analysis of nonlinear systems, the damping effects should be considered.

This paper presents a new method to derive a group of Lyapunov functions to reflect the damping effects by considering the integral relationships of the system governing equations.

Key words : Nonlinear System; Stability Analysis; Direct Method; Lyapunov Function; Energy Function

I. Introduction

In the operation of Nonlinear Systems, the stability problem is the first issue to be solved. Many authors contributed to developing the nonlinear stability theory, yielding Popov theorem, and the ultimate confinement theorem for Lure-Type nonlinear systems.

Most of the theorems of nonlinear stability is based on the Lyapunov stability theory. The Lyapunov function method is most well-known and provides precise and rigorous theoretical backgrounds. This method has a great merit that it is possible to tell the future system stability by only the present state. However, this method has not been widely used since there is no general method to find appropriate Lyapunov functions. One of the popular methods to find a Lyapunov function is using the energy function of the system. Most of the nonlinear systems have no global Lyapunov function, and thus local Lyapunov functions are generally used to determine a local stability around a certain singular solution in concern. In this case, system damping should be considered as an important factor to determine the local stability. However, there has been no attempt to reflect the damping effects in the Lyapunov function. This paper presents a new method to derive a group of Lyapunov functions to reflect the damping effects by considering the integral relationships of the system governing equations.

II. Local Stability analysis by using Energy functions

Given a nonlinear system, it is relatively ease to find an energy function E, which satisfy $\frac{dE}{dt} \le 0$ for all time. Assume that we are concerned with the system stability around a certain singular solution x^e. The local stability around x^e can be determined by the following Lemma, which is only a sufficient condition.

Lemma : If there is a convex region R_s around x^e and the energy function $E(x)$ is a convex function in R_s, then the system is locally stable around x^e.

The above Lemma can be widely applied to local stability analysis of nonlinear systems if some appropriate energy function is provided. Most of the conventional studies have adopted the dynamic energy function which is given by the sum of kinetic and potential energies, and the conventional all the system losses decrease the system energy with $\frac{dE}{dt} \le 0$. Therefore, the Lyapunov method usually neglects the system losses. The Lyapunov condition is just sufficient condition, and there is some possibility to find a more wide stable region by considering the system damping.

In this study, it will be shown that a group of local Lyapunov can be derived to reflect the damping effects by considering the integral relationship of the system governing equations.

III. Derivation of a Group of Lyapunov function by considering system losses

There is no other thumb rule to find a Lyapunov function than investigating case-by-case. The most popular method to find a Lyapunov function is taking the energy function as a Lyapunov function. This study will also show how to derive a group of energy-related Lyapunov functions by considering system losses with the following sample R-L-C system.

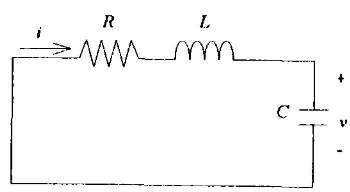


Fig.1 RLC Circuit

The state equation for the above system is given by

$$\dot{v} = \frac{1}{C} i \qquad \text{-------------------(1.a)}$$

$$\dot{i} = -\frac{R}{L} i - \frac{1}{L} v \qquad \text{-----------------------(1.b)}$$

The energy function for this system is given by

$$E = \frac{1}{2} L i^2 + \frac{1}{2} C v^2 \qquad \text{----------------------(2)}$$

Then, it can be easily shown that the time derivative energy function is given by

$$\frac{dE}{dt} = -R i^2 \qquad \text{---------------------(3)}$$

This energy function can be taken as a Lyapunov function. However, the energy function E in Eq.(2) does not reflect the system loss Ri^2. We can derive some other type of Lyapunov function by using integral relations of the system equation.

Equation (1.b) can be rewritten as follows:

$$L\dot{i} = -Ri - v \qquad \text{----------------------(4)}$$

$$L\dot{i}^2 = -Ri\dot{i} - v\dot{i}$$
$$= -Ri\dot{i} - vC\ddot{v} \qquad \text{----------------(5)}$$

Integrating both sides of the above equation gives

$$L\int \dot{i}^2 dt = -\frac{1}{2} R i^2 - C\int v\ddot{v} dt \qquad \text{-----------------(6)}$$

The last term of Eq.(6) can be rewritten as

$$\int v\ddot{v} dt = v\dot{v} - \int \dot{v}^2 dt$$
$$= Cvi - \int \dot{v}^2 dt \qquad \text{---------------- (7)}$$

By substituting Eq.(7) in Eq.(6), we can obtain

$$\int \dot{i}^2 dt = -\frac{1}{2}\frac{R}{L} i^2 - \frac{1}{L} vi + \frac{C}{L}\int \dot{v}^2 dt \qquad \text{-------------- (8)}$$

For the given system, we have the following energy conservation equation.

$$\frac{1}{2} L i^2 + \frac{1}{2} C v^2 + \int R i^2 dt = K \qquad \text{-------------------(9)}$$

where K is a constant number.

The substitution of Eq.(8) into Eq.(9) yields

$$\frac{1}{2}(L - \frac{R^2}{L}) i^2 - \frac{R}{L} vi + \frac{1}{2} cv^2 + \frac{CR}{L}\int \dot{v}^2 dt = K \qquad \text{-----(10)}$$

If the system parameters satisfy the following condition

$$D = \left(\frac{R}{L}\right)^2 - \left(L - \frac{R^2}{L}\right) C \le 0, \qquad \text{---------(11)}$$

we can get another Lyapunov function as follows:

$$E_1 = \frac{1}{2}\left(L - \frac{R^2}{L}\right) i^2 - \frac{R}{L} vi + \frac{1}{2} C v^2 \qquad \text{------------(12)}$$

Here, it is noted that function E_1 is convex if inequality (11) holds.

The time-derivative of E_1 can be given by

$$\frac{dE_1}{dt} = -\frac{CR}{L}\dot{v}^2 \qquad \text{---------------(13)}$$

This relation can be easily proven either by the conventional method or by differentiating the transformed energy conservation rule in Eq.(10).

By examining Eqs.(12) and (13), we can see that a new energy function E_1 can be another type of energy-related Lyapunov function if the system parameters satisfy the condition (11).

On the other hand, we can derive a group of energy functions by splitting the damping terms. Eq.(9) can be rewritten as follows :

$$\frac{1}{2} L i^2 + \frac{1}{2} C v^2 + \lambda\int R i^2 dt + (1-\lambda)\int R i^2 dt = K \qquad \text{------(14)}$$

with $0 \le \lambda \le 1$

By replacing the first damping-related term by Eq.(14) and rearranging it, we can obtain the following equation.

$$\frac{1}{2}\left(L - \frac{R^2}{L}\lambda\right) i^2 - \frac{R}{L}\lambda vi + \frac{1}{2} C v^2 + \int\left[\frac{RC}{L}\lambda \dot{v}^2 + (1-\lambda) R i^2\right] dt = K$$

--------------(15)

The above equation can be called as variants of the energy conservation law in Eq.(9) with all $\lambda \in [0,1]$.

By examining Eq.(15), we can obtain a group of Lyapunov functions as follows :

$$L_\lambda(i,v) = \frac{1}{2}\left(L - \frac{R^2}{L}\lambda\right) i^2 - \frac{R}{L}\lambda vi + \frac{1}{2} C v^2 \qquad \text{----------(16)}$$

with all $\lambda \in [0,1]$ which satisfies

$$D = \left(\frac{R}{L}\right)^2 \lambda^2 - \left(L - \frac{R^2}{L}\lambda\right) C$$

$$= \frac{R^2}{L^2}\lambda^2 + \frac{R^2 C}{L}\lambda - LC \le 0 \qquad \text{-------------(17)}$$

The time derivative of Lyapunov function L_λ is given by

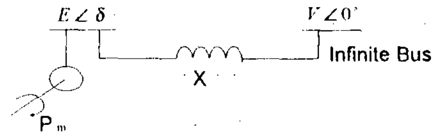
$$\frac{d}{dt} L_\lambda(i,v) = -\frac{RC}{L}\lambda v^2 - (1-\lambda) R i^2 \le 0 \qquad \text{---------(18)}$$

Condition (17) guarantees that $L_\lambda(i,v)$ is a convex function, and Eq.(18) shows the seminegativeness of the time derivative of the Lyapunov function.

The above typical example shows that we can derive a group of Lyapunov function for a simple RLC circuit. This method can be applied to harmonic oscillator problems such as pendulum oscillators and swing equations of power system with constant damping.

IV. Illustrative Example

Consider a one-machine infinite bus system with a pure reactive line as shown in Fig.2



M : Generator inertia
D : Generator damping coefficient

Fig. 2

The swing equation of the above system is given by

$$\dot{\delta} = \omega \qquad \text{-----------(19.a)}$$

$$M\dot{\omega} + D\omega = P_m - \frac{EV\sin\delta}{X_t} \qquad \text{-----------(19.b)}$$

where M : Generator Inertia
D : Generator Damping Coefficient
P_m : Mechanical Input Power [MW]

For stability analysis of the system, the mechanical input P_m is considered to be constant, and the generator internal voltage E is also assumed to be well governed to be constant. Under these assumptions, the conventional system energy is given by

$$E = \frac{1}{2}M\omega^2 + \frac{EV(\cos\delta_s - \cos\delta)}{X_t} - P_m(\delta - \delta_s) \qquad \text{-------(20)}$$

where δ_s is a singular solution of the swing equations,

$$\text{i.e.}\quad \delta_s = \sin^{-1}\left(\frac{X_t P_m}{EV}\right) \qquad \text{---------------(21)}$$

The time derivative of the energy function E is given by

$$\frac{dE}{dt} = -D\omega^2 \qquad \text{------------------(22)}$$

The energy conservation can be described as follows :

$$\frac{1}{2}M\omega^2 + \frac{EV(\cos\delta_s - \cos\delta)}{X_t} - P_m(\delta - \delta_s) + \int_0^t D\omega^2 dt = K \qquad \text{-----------------(23)}$$

The energy function E has an energy well around the singular solution as shown in Fig. 3.

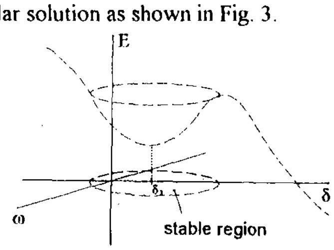


Fig. 3 Energy well of function E

The energy function E directly gives the stable region S shadowed in Fig.3. However, energy function E does not reflect the damping effects. When the damping coefficient is significantly large, the stable region should be extended. In order to reflect the damping effects, we can derive an integral equation from Eq.(19.b). First multiplying the both sides of Eq.(19.b) by $\Delta\delta$ and integrating with respect to time t, we get

$$\int_0^t M\dot{\omega}\Delta\delta dt + \int_0^t D\omega\Delta\delta dt = \int_0^t (P_m - P_e)\Delta\delta dt \qquad \text{--------(24.a)}$$

$$\text{where } \Delta\delta = \delta - \delta_s \qquad \text{---------(24.b)}$$

In the above equation, the angular velocity ω can be represented as follows:

$$\omega = \frac{d\delta}{dt} = \frac{d\Delta\delta}{dt} \qquad \text{----------------(25)}$$

Substitution of Eq.(25) in Eq.(24.a) gives

$$\int_0^t M\Delta\ddot{\delta}\Delta\delta dt + \int_0^t D\Delta\dot{\delta}\Delta\delta dt = \int_0^t (P_m - P_e)\Delta\delta dt \qquad \text{------(26.a)}$$

$$\text{where } P_e = \frac{EV\sin\delta}{X_t} \qquad \text{------------(26.b)}$$

Here, the first term in Eq.(26.a) can be rewritten as follows:

$$\int_0^t M\Delta\ddot{\delta}\Delta\delta dt = M\Delta\dot{\delta}\Delta\delta\Big|_0^t - \int_0^t M\Delta\dot{\delta}^2 dt$$

$$= M\omega\Delta\delta - M\int_0^t \omega^2 dt \qquad \text{---------(27)}$$

Since the second term of Eq.(26.a) can be directly integrated, the substitution of Eq.(27) into Eq.(26.a) gives.

$$M\omega\Delta\delta - M\int_0^t \omega^2 dt + \frac{1}{2}D\Delta\delta^2 = \int_0^t (P_m - P_e)\Delta\delta dt$$

From the above equation, we can obtain the following integral relationship:

$$\int_0^t \omega^2 dt = \omega\Delta\delta + \frac{1}{2}\frac{D}{M}\Delta\delta^2 - \frac{1}{M}\int_0^t (P_m - P_e)\Delta\delta dt \qquad \text{--(28)}$$

By using this relationship, we can derive a group of energy functions from Eq.(23)

$$E_\lambda(\omega,\delta) = \frac{1}{2}M\omega^2 + \frac{EV(\cos\delta_s - \cos\delta)}{X_t} - D\lambda\omega\Delta\delta - \frac{1}{2}\frac{D^2}{M}\lambda\Delta\delta^2 - P_m(\delta - \delta_s) \qquad \text{---------(29)}$$

with $0 \le \lambda \le 1$

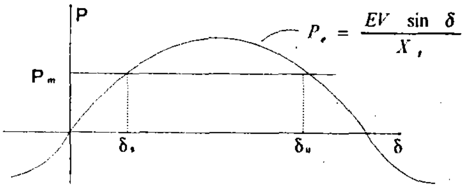
The time-derivative of E_λ is given by

$$\frac{dE_\lambda}{dt} = -(1-\lambda)D\omega^2 + \frac{\lambda}{M}(P_m - P_e)\Delta\delta \le 0 \quad \text{for all time} \qquad \text{------(30)}$$

In the above equation, the last term has seminegativeness since singular solution δ_s can be determined by the following power curve in Fig.4, which shows that

$$(P_m - P_e)\Delta\delta \le 0 \qquad \text{for all time.}$$

As a result, the seminegativeness of the time-derivative of E_λ can be always guaranteed for all $\lambda \in [0,1]$



If $\Delta\delta_s > 0$, then $P_m - P_e > 0$
If $\Delta\delta_s < 0$, then $P_m - P_e < 0$

Fig.4 Power Curve

Now, we will examine the local convexness of the energy function E_λ around the singular solution. E_λ can be approximated around $\delta = \delta_s$ as follows:

$$E_\lambda \cong \frac{1}{2}M\omega^2 + \frac{EV\sin\delta_s}{X_t}\Delta\delta + \frac{1}{2}\frac{EV\cos\delta_s}{X_t}\Delta\delta^2 - D\lambda\omega\Delta\delta - \frac{1}{2}\frac{D^2}{M}\lambda\Delta\delta^2 - P_m\Delta\delta$$
$$= \frac{1}{2}M\omega^2 - D\lambda\omega\Delta\delta + \frac{1}{2}\left(\frac{EV\cos\delta_s}{X_t} - \frac{D^2}{M}\lambda\right)\Delta\delta^2 \quad \text{------(31)}$$

In the above equation, the following relationship is used,

$$P_m = \frac{EV\sin\delta_s}{X_t} \quad \text{------(32)}$$

In the Eq. (31), E_λ can be convex around X_s if

$$(D\lambda)^2 - M\left(\frac{EV\cos\delta_s}{X_t} - \frac{D^2}{M}\lambda\right) \le 0 \quad \text{------(33)}$$

As a result, it can be concluded that, for any $\lambda \in [0,1]$ which satisfies the condition (33), Energy function E_λ can be used as a local Lyapunov function.

By selecting an appropriate λ, we can establish a Lyapunov function to reflect the damping effects. In order to reflect the damping effects as far as possible, we should select the largest λ which satisfies the condition (33). In that case, we

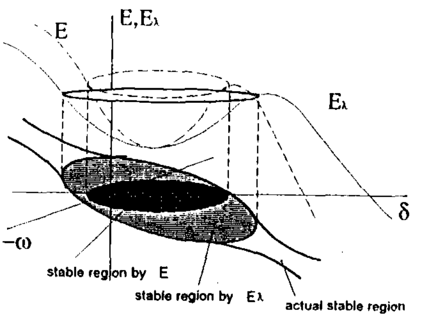


Fig. 5 Extended stable region by E_λ

can easily check that the stable region is extended with a slight rotation of the ellipse as shown in Fig. 5

V. CONCLUSIONS

This paper presents a new method to derive a group of energy-related Lyapunov functions to reflect the damp effects by considering the system governing equations. For stability analysis of nonlinear systems, the local stability has been discussed. In order to reflect damping effects for a system with significantly large damping coefficients, a systematical approach has been developed to convert some part of damping loss into some appropriate system energy terms. An illustrative example has shown that the proposed method can be well applied to harmonic oscillator problems, and that, by using the Lyapunov function reflecting damping effects, the accuracy of the direct method can be improved.

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