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LOW-ORDER H_∞ CONTROLLERS WITH THE MIXED SENSITIVITY PROBLEMS

Run–De Qi and Teruo Tsuji
Department of Electrical Engineering
Kyushu Institute of Technology

Abstract

This paper presents a simple methodology for reducing the order of H_∞ controllers in the mixed sensitivity control problems. The key point of this methodology is to transform the generalized plant expression to new one, where the control object and the weighting functions for the sensitivity function may have some poles on the imaginary axis. So that, this methodology makes it possible to use the standard method to slove the general H_∞ design problems about the mixed sensitivity problems, even for a servo system or a oscillatory system. We derive that the order of H_∞ controllers designed by this methodology may be reduced to n_p where n_p is the order of the denominator of the control object. It is clear that n_p is lower than $n_p + n_s$, which is the order of H_∞ controllers obtained by the ordinary H_∞ design method up to now, where n_s is the order of the denominator of the weighting function for sensitivity. Finally, a numerical example is given to illustrate the results.

Keywords: H_∞ controllers, Low-order stabilizing controllers, Mixed sensitivity problem.

Notations

$\mathbf{R}^{n\times m}$: the set of $n \times m$ proper real–rational matrices.
$\mathbf{C}^0$: the imaginary axis.
$\mathbf{C}^+$: the open right half plane.
$\bar{\mathbf{C}}^+$: the closed right half plane.
$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D.$

1 Introduction

In this paper, about the H_∞ controller design of single input single output systems, we consider a control system in Figure 1, so-called the two block mixed sensitivity H_∞ design problem, where $P(s)$ is the control object, $W_s(s)$ and $W_t(s)$ are weighting functions for the sensitivity function S ($S = (I + PK)^{-1}$) and the complementary sensitivity function T ($T = PKS$) respectively. The signals are as follows: $\omega(s)$ is the disturbance vector; $u(s)$ is the control input vector; $z(s)$ is the controlled output vector; and $y(s)$ is the measured output vector.

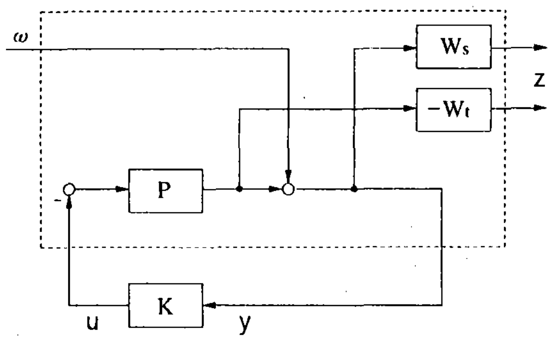

Fig.1. The mixed sensitivity H_∞ problem.

The 'H_∞ control problem' is then to choose a controller, $K(s)$, that makes the closed-loop system internally stable and minimizes the H_∞-norm of the transfer function from $\omega(s)$ to $z(s)$. In fact, we will consider the problem of finding a stabilizing K such that

$$\|LFT(G,K)\|_\infty < \gamma \qquad \gamma \in \mathbf{R}^+ \tag{1}$$

where[7]

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W_s \\ 0 \end{bmatrix} & \begin{bmatrix} -W_sP \\ W_tP \end{bmatrix} \\ [I] & [-P] \end{bmatrix} \tag{2}$$

and

$$LFT(G,K) = \begin{bmatrix} W_s \\ 0 \end{bmatrix} + \begin{bmatrix} -W_sP \\ W_tP \end{bmatrix} K(I+PK)^{-1}I = \begin{bmatrix} W_sS \\ W_tT \end{bmatrix} \tag{3}$$

Note that $-G_{22}(s)$ is the control object. So the internal stable problem is to choose a controller, $K(s)$, that makes the $P(s)$ internally stable. Since $S+T=1$, it connot be satisfied simultaneously to make gains ($\| S(j\omega) \|$ and $\| T(j\omega) \|$) as low as possible together. We try to obtain the low sensitivity on the low frequency band and consider the robust stability firstly on the high frequency band by choosing suitable weighting functions separately, $W_s(s)$ and $W_t(s)$. If such a $K(s)$ exists, we say that the H_∞ problem is solvable. The state-space realizations of $P(s)$, $W_s(s)$ and $W_t(s)P(s)$ will be denoted as

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right] \quad (4)$$

$$W_s(s) = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right] \quad (5)$$

$$W_t(s)P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_t & D_t \end{array}\right] \quad (6)$$

where $A_p \in \mathbf{R}^{n_p \times n_p}$ and $A_s \in \mathbf{R}^{n_s \times n_s}$, then the state-space representation of the generalized plant $G(s)$ in (2) is [7]

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|cc} \begin{array}{cc} A_s & B_sC_p \\ 0 & A_p \end{array} & \begin{bmatrix} B_s \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -B_p \end{bmatrix} \\ \hline \begin{bmatrix} C_s & D_sC_p \\ 0 & -C_t \end{bmatrix} & \begin{bmatrix} D_s \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ D_t \end{bmatrix} \\ \begin{bmatrix} 0 & C_p \end{bmatrix} & [I] & [0] \end{array}\right] \quad (7)$$

where $A \in \mathbf{R}^{(n_s+n_p)\times(n_s+n_p)}$.

The following assumptions are made to ensure well-posedness and the existence of a controller [1].

A1. (A, B_2, C_2) is stabilizable and detectable.

A2. D_{12} has full column rank and D_{21} has full row rank.

A3. D_{12} and D_{21} are transformed into $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$ by a scaling of u and y, together with a unitary transformation of ω and z. D_{11} is partitioned compatibility with D_{12} and D_{21} as $D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$.

A4. For all $\forall\, \omega \in \mathbf{R}$, $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank.

A5. For all $\forall\, \omega \in \mathbf{R}$, $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank.

Recently, a theory of optimal H_∞ design has been widely developed. Among all tools available for such design, the mathod of Glover and Doyle [1] is widely acclaimed which is called the standard H_∞ control design. In the ordinary H_∞ control, the control object $P(s)$ and weighting functions are not to have poles in $\mathbf{C}^0$, so that the assumptions, A1 and A5 are satisfied and the order of the designed H_∞ controller is equal to $n_p + n_s$ when the H_∞ problem is solvable. The order of this controller is higher usually. But, in some case, it is necessary for a system design that the control object or weighting functions have some poles in $\mathbf{C}^0$, in particullar, when we want to design a servo system. In recent years, Zhou and Khargonekar [2], Sampei, Mita, and Nakamichi [3], Koide, Hara, and Kondo [4] have presented the methods to solve this problem, which introduce a sufficiently small positive constants ε in the process of solving the H_∞ prollem. Since these methods depend on the value ε, we have to work very hard to choos ε suitably. Liu and Mita [5] have treated the same problem, but they do not use the standard H_∞ control design. All of the methods [2] to [5] also give that the order of the designed H_∞ controller is equal to $n_p + n_s$. Otherhand, Gu and Choi [6] have studied the problem of reducing the order of H_∞ controllers, but the design will become difficuld when the control object or weighting functions have some poles in $\mathbf{C}^0$.

The purpose of this paper is to present a simple and effective methodology for the H_∞ control problem of the two block mixed sensitivity, where the control object or weighting functions for the senstivity function may have some poles in $\mathbf{C}^0$. It can directly apply the standard H_∞ control design [1] to slove this problem and will reduce the order of the designed H_∞ controller into n_p. Our methodology can be also applied to the H_∞ design of $MIMO$ by using the decoupling control to transform it into an H_∞ design of $SISO$. As for the design of a servo system, we can treat it by multiplying a integrator to the control object beforehand.

2 Low-order H_∞ controllers

Cosider a general control object, $P(s)$ described by a transfer function

$$P(s) = \frac{k_p(s+b_1)(s+b_2)\dots(s+b_{m_p})}{(s+a_1)(s+a_2)\dots(s+a_{n_p})}, \quad (8)$$

where the transfer function is strictly proper and has no the transmission zeros in $\mathbf{C}^0$. We define a set $\mathbf{C}^-_{\max}$ for the poles $-a_i (i = 1, 2, \dots n_p)$ of $P(s)$ as follows:

$$\mathbf{C}^-_{\max} =: \{-a_j \mid R_e(-a_j) = max(R_e(-a_i)),\; a_j = a_i,\; a_i \in \bar{\mathbf{C}}^+\}$$

Remark: We assume that $\mathbf{C}^-_{\max} \neq \phi$ namely $\mathbf{C}^-_{\max}$ is not a null set in the following discussion.

By this definition, we choose that all the poles of the weighting function $W_s(s)$ are equal to all the elements of $\mathbf{C}^-_{\max}$. Assume the state-space representation of $W_s(s)$ be given in (5) and (A_s, C_s) be detectable.

Physical meaning: When we design a servo system, the origin belongs to $\mathbf{C}^-_{\max}$. The gradient of the gain plot of $W_s(s)^{-1}$ is coincident with that of the gain plot of $S(s)$ on the low frequency band. Therefore, the gain of $S(s)$ can be reduced graetly on the low frequency band, namely, the low sensitivity can be realized easily.

We divide $P(s)$ to

$$P(s) = P_1(s)P_2(s) \quad (9)$$

and consider the following two cases.

1). $\mathbf{C}^-_{\max} \subset \mathbf{C}^0$:

It means that $P(s)$ has poles on the imaginary axis. We choose that $P_1(s)$ has all the poles of $P(s)$ in $\mathbf{C}^-_{\max}$, and has no any finite transmission zeros, namely the numerator of $P_1(s)$ is a constant. Therefore $P_2(s)$ has neither poles in $\mathbf{C}^-_{\max}$ nor transmission zeros in $\mathbf{C}^0$ by the assumption for $P(s)$. Let $P(s)$ have r poles in $\mathbf{C}^0$. Take a polynominal

$$M(s) = (s+d_1)(s+d_2)\dots(s+d_r) \qquad (10)$$
$$d_i \in \mathbf{C}^+,\ i = 1,2,\dots,r$$

to change the division of $P(s)$ as

$$P(s) = M(s)P_1(s) \times \frac{1}{M(s)}P_2(s) \qquad (11)$$

where $d_i \in \mathbf{C}^+$ means stable poles or zeros.

Define

$$P_{m1}(s) := M(s)P_1(s) = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_{pm1} & D_{pm1}\end{array}\right] \qquad (12)$$

$$P_{m2}(s) := \frac{1}{M(s)}P_2(s) = \left[\begin{array}{c|c} A_{pm2} & B_{pm2} \\ \hline C_{pm2} & 0\end{array}\right] \qquad (13)$$

then

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0\end{array}\right] = \left[\begin{array}{cc|c} A_s & B_sC_{pm2} & 0 \\ 0 & A_{pm2} & B_{pm2} \\ \hline C_{pm1} & D_{pm1}C_{pm2} & 0\end{array}\right] \qquad (14)$$

It is clear that $D_{pm1} \neq 0$ and $n_{pm2} = n_p$, where n_{pm2} is the order of the denominator of $P_{m2}(s)$.

2). $\mathbf{C}^-_{\max} \not\subset \mathbf{C}^0$:

It means that $P(s)$ has no any poles on the imaginary axis. Let

$$P_{m1}(s) = P_1(s) = 1 \qquad (15)$$
$$P_{m2}(s) = P_2(s) = P(s) \qquad (16)$$

The state-space representation of $P_{m1}(s)$ will also be given in (12) but $C_{pm1} = 0$ and $D_{pm1} = 1$, then

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0\end{array}\right] = \left[\begin{array}{c|c} A_{pm2} & B_{pm2} \\ \hline C_{pm2} & 0\end{array}\right] \qquad (17)$$

From (9) to (16), we have the next lemma.

Lemma 2.1. As for the mixed sensitivity H_∞ control problem in Figure 1, the generalized plant G can be given by

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W_s \\ 0 \end{bmatrix} & \begin{bmatrix} -W_sP_{m2} \\ W_tP_{m2}\end{bmatrix} \\ [P_{m1}] & [-P] \end{bmatrix} \qquad (18)$$

Proof. By (11) – (16),

$$\begin{aligned} LFT(G,K) &= \begin{bmatrix} W_s \\ 0 \end{bmatrix} + \begin{bmatrix} -W_sP_{m2} \\ W_tP_{m2}\end{bmatrix} K(I+PK)^{-1}P_{m1} \\ &= \begin{bmatrix} W_s \\ 0 \end{bmatrix} + \begin{bmatrix} -W_sP \\ W_tP\end{bmatrix} K(I+PK)^{-1} \\ &= \begin{bmatrix} W_sS \\ W_tT \end{bmatrix} \quad \Box \end{aligned}$$

Now we choose $W_t(s)$ so that $W_t(\infty)P_{m2}(\infty) \neq 0$, namely, $W_t(s)P_{m2}(s)$ has no transmission zeros in the infinity. To simplify discussion, we let $W_t(s)P_{m2}(s)$ be given by a minimal realization as follows with choosing matrices C_t and D_t suitably,

$$W_t(s)P_{m2}(s) = \left[\begin{array}{c|c} A_{pm2} & B_{pm2} \\ \hline C_t & D_t\end{array}\right] \qquad (19)$$

where $D_t \neq 0$.

Theorem 2.2. The generalized plant $G(s)$ in (18) can be described by a state-space realization as follows

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|c|c} A_s & B_sC_{pm2} & \begin{bmatrix} B_s \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -B_{pm2} \end{bmatrix} \\ 0 & A_{pm2} & & \\ \hline \begin{bmatrix} C_s & D_sC_{pm2} \\ 0 & -C_t \end{bmatrix} & & \begin{bmatrix} D_s \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ D_t \end{bmatrix} \\ [C_{pm1} & D_{pm1}C_{pm2}] & [D_{pm1}] & [0] \end{array}\right] \qquad (20)$$

where $A \in \mathbf{R}^{(n_p+r)\times(n_p+r)}$.

Proof.

$$\begin{aligned} G_{11}(s) &= \begin{bmatrix} D_s \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} C_s & D_sC_{pm2} \\ 0 & -C_t \end{bmatrix} \begin{bmatrix} sI - A_s & -B_sC_{pm2} \\ 0 & sI - A_{pm2}\end{bmatrix}^{-1} \begin{bmatrix} B_s \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} D_s + C_s(sI-A_s)^{-1}B_s \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} W_s(s) \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} G_{12}(s) &= \begin{bmatrix} 0 \\ D_t \end{bmatrix} + \begin{bmatrix} C_s & D_sC_{pm2} \\ 0 & -C_t \end{bmatrix} \times \\ &\quad \begin{bmatrix} sI - A_s & -B_sC_{pm2} \\ 0 & sI - A_{pm2}\end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -B_{pm2} \end{bmatrix} \\ &= \begin{bmatrix} (-D_s - C_s(sI-A_s)^{-1}B_s)P_{pm2}(s) \\ D_t + C_t(sI-A_{pm2})^{-1}B_{pm2} \end{bmatrix} \\ &= \begin{bmatrix} -W_s(s)P_{pm2}(s) \\ W_t(s)P_{pm2}(s) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} G_{21}(s) &= D_{pm1} + \begin{bmatrix} C_{pm1} & D_{pm1}C_{pm2} \end{bmatrix} \times \\ &\quad \begin{bmatrix} sI - A_s & -B_sC_{pm2} \\ 0 & sI - A_{pm2}\end{bmatrix}^{-1} \begin{bmatrix} B_s \\ 0 \end{bmatrix} \\ &= D_{pm1} + C_{pm1}(sI-A_s)^{-1}B_s \\ &= P_{pm1}(s) \end{aligned}$$

$$\begin{aligned} G_{22}(s) &= 0 + \begin{bmatrix} C_{pm1} & D_{pm1}C_{pm2} \end{bmatrix} \times \\ &\quad \begin{bmatrix} sI - A_s & -B_sC_{pm2} \\ 0 & sI - A_{pm2}\end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -B_{pm2} \end{bmatrix} \\ &= -D_{pm1}C_{pm2}(sI-A_{pm2})^{-1}B_{pm2} \\ &- C_{pm1}(sI-A_s)^{-1}B_sC_{pm2}(sI-A_{pm2})^{-1}B_{pm2} \\ &= -P_{pm1}(s)P_{pm2}(s) \\ &= -P(s) \quad \Box \end{aligned}$$

For the state-space realization of $G(s)$ in (20), we will check the assumptions A1-A5 made in the standard H_∞ design method.

About the assumption A1, the next lemma is given.

Lemma 2.3. In the state-space realization in (20), (A, B_2, C_2) is stabilizable and detectable if and only if (A_p, B_p, C_p) in (4) is stabilizable and detectable.

Proof.

1^0). $\mathbf{C}^-_{max} \not\subset \mathbf{C}^0$, by (15) to (17),

$$\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_s & B_sC_p & 0 \\ 0 & A_p & -B_p \\ \hline 0 & C_p & 0 \end{array}\right]$$

Because $W_s(s)$ has no poles in $\bar{\mathbf{C}}^+$, therefore it is clear,
(A, B_2, C_2) is stabilizable and detectable
$\Longleftrightarrow (A_p, B_p, C_p)$ is stabilizable and detectable.

2^0). $\mathbf{C}^-_{max} \subset \mathbf{C}^0$, by (10) and (14),

$$\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_s & B_sC_{pm2} & 0 \\ 0 & A_{pm2} & -B_{pm2} \\ \hline C_{pm1} & D_{pm1}C_{pm2} & 0 \end{array}\right] = \left[\begin{array}{c|c} A_p & -B_p \\ \hline C_p & 0 \end{array}\right]$$

and the pole/zero cancellation model $\frac{M(s)}{M(s)}$ is stable function, therefore
(A, B_2, C_2) is stabilizable and detectable
$\Longleftrightarrow (A_p, B_p, C_p)$ is stabilizable and detectable. □

By the definition of $P_{m1}(s)$ in (12), (15) and the choice of $W_t(s)$ in (19), it is clear that

$$D_{pm1} \neq 0 \;\; and \;\; D_t \neq 0$$

Therefore, the assumption A2 is satisfied for the state-space realization in (20). Take a scaling of u and y, together with a unitary transformation of ω and z, the assumption A3 is satisfied for (20) too.

A suitable choice of $W_t(s)$ enables us to assume without loss of generality that $W_t(s)P_{m2}(s)$ has no transmission zeros in $\mathbf{C}^0$. The assumption A4 will be satisfied by the next lemma.

Lemma 2.4. In the state-space realization in (20), for all $\forall\, \omega \in \mathbf{R}$,

$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full column rank.

Proof. By (20),

$$rank \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = rank \begin{bmatrix} A_s - j\omega I & B_sC_{pm2} & 0 \\ 0 & A_{pm2} - j\omega I & -B_{pm2} \\ C_s & D_sC_{pm2} & 0 \\ 0 & -C_t & D_t \end{bmatrix} = rank \begin{bmatrix} A_s - j\omega I \\ C_s \end{bmatrix} + rank \begin{bmatrix} A_{pm2} - j\omega I & -B_{pm2} \\ -C_t & D_t \end{bmatrix}$$

Because (A_s, C_s) is detectable, then for all $\forall\, \omega \in \mathbf{R}$,

$$\begin{bmatrix} A_s - j\omega I \\ C_s \end{bmatrix}$$

has full column rank. By the assumption that $W_t(s)$ $P_{m2}(s)$ has no transmission zeros in $\mathbf{C}^0$, then for all $\forall\, \omega \in \mathbf{R}$,

$$\begin{bmatrix} A_{pm2} - j\omega I & -B_{pm2} \\ -C_t & D_t \end{bmatrix}$$

has full rank. □

In the same way, the assumption A5 will be satisfied by the next lemma.

Lemma 2.5. In the state-space realization in (20), for all $\forall\, \omega \in \mathbf{R}$,

$$\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has full row rank.

Proof. By (20),

$$rank \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = rank \begin{bmatrix} A_s - j\omega I & B_sC_{pm2} & B_s \\ 0 & A_{pm2} - j\omega I & 0 \\ C_{pm1} & D_{pm1}C_{pm2} & D_{pm1} \end{bmatrix} = rank \begin{bmatrix} A_s - j\omega I & B_s \\ C_{pm1} & D_{pm1} \end{bmatrix} + rank \begin{bmatrix} A_{pm2} - j\omega I \end{bmatrix}$$

Because $P_{m2}(s)$ has no poles in $\mathbf{C}^0$, then for all $\forall\, \omega \in \mathbf{R}$, $[A_{pm2} - j\omega I]$ has full rank. By the choice of $M(s)$ in (10) and the definition of $P_{m1}(s)$ in (12), (15), $P_{m1}(s)$ has no transmission zeros in $\mathbf{C}^0$, then for all $\forall\, \omega \in \mathbf{R}$,

$$\begin{bmatrix} A_s - j\omega I & B_s \\ C_{pm1} & D_{pm1} \end{bmatrix}$$

has full rank. □

For the general control object $P(s)$ given by (8), we can derive the next theorem, which is about the order n_k of the denominator of the controller designed by our mathodology.

Theorem 2.6. For the general control object $P(s)$ given by (8), we have derived that the state-space representation of the generalized plant $G(s)$ can be described by (20) as the mixed sensitivity H_∞ control problem. If the H_∞ problem is solvable, the order n_k of the denominator of the controller designed will be equal to n_p, where the controller is the central solution of the H_∞ control problem.

Proof. By (20), $A \in \mathbf{R}^{(n_p+r)\times(n_p+r)}$, we know that the order n_k of the designed H_∞ controller will be equal to $n_p + r$, but

1^0). $\mathbf{C}^-_{max} \subset \mathbf{C}^0$, the order n_m of the polynominal $M(s)$ is equal to r. The transmission zeros and poles of the central solution have the stable poles of $W_t(s)P_{m2}(s)$ and the transmission zeros of $P_{m1}(s)$ respectively [9]. Namely, the central solution has the same r stable poles and transmission zeros which correspond to the zeros of $M(s)$. By cancelling these r poles and zeros, the order n_k of the designed H_∞ controller can be reduced to n_p.

2^0). $\mathbf{C}^-_{\max} \not\subset \mathbf{C}^0$, the transmission zeros and poles of the central solution have the stable poles of $P_{m2}(s)$ and $W_s(s)$ respectively [9]. By the choice of $W_s(s)$, the number of the poles of $W_s(s)$ is r and they all belong to the stable poles of $P_{m2}(s)$. Therefore, the order n_k of the designed H_∞ controller can be reduced to n_p by cancelling these r stable poles and zeros which are equal to the poles of $W_s(s)$. □

Remark: When the zeros of $M(s)$ are equal to a part of the stable transmission zeros of $P(s)$, the size of A matrix of the generalized plant $G(s)$ will decrease, but these lemma and theorem above also hold with a simple variation. And the designed controller as the central solution of the H_∞ control problem will have the poles which are equal to this part of the stable transmission zeros of $P(s)$.

3 A numerical example

Let us consider a example as follows in order to illustrate the application of our methodology.

$$P(s) = \frac{1}{s(s^2+4)} \tag{21}$$

where the control object has poles $0, \pm j2$. So we define $\mathbf{C}^-_{\max} = \{0, \pm j2\}$ and take the polynominal $M(s)$ as

$$M(s) = (s+1)(s+3)(s+5) \tag{22}$$

Divide $P(s)$ to

$$P_{m1}(s) = \frac{(s+1)(s+3)(s+5)}{s(s^2+4)} \tag{23}$$

$$P_{m2}(s) = \frac{1}{(s+1)(s+3)(s+5)} \tag{24}$$

The weightings functions are chosen as

$$W_s(s) = \frac{\rho}{s(s^2+4)} \tag{25}$$

$$W_t(s) = \frac{(s+\alpha)^3}{\alpha^3\beta} \tag{26}$$

According to the state-space realization of $W_t(s)P_{m2}(s)$ given by (19), we can obtain C_t and D_t as follows

$$\begin{aligned} C_t &= \begin{bmatrix} C_{t1} & C_{t2} & C_{t3} \end{bmatrix} \\ &= \frac{1}{\alpha^3\beta}\begin{bmatrix} \alpha^3-15 & 3\alpha^2-23 & 3\alpha-9 \end{bmatrix} \end{aligned} \tag{27}$$

$$D_t = \frac{1}{\alpha^3\beta} \tag{28}$$

By (20), we can obtain the state-space description of the generalized plant $G(s)$ as

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c|c} \begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -15 & -23 & -9 \end{matrix} & \begin{bmatrix} 0\\0\\1\\0\\0\\0 \end{bmatrix} & \begin{bmatrix} 0\\0\\0\\0\\0\\-1 \end{bmatrix} \\ \hline \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_{t1} & -C_{t2} & -C_{t3} \end{bmatrix} & \begin{bmatrix} 0\\0 \end{bmatrix} & \begin{bmatrix} 0\\D_{t1} \end{bmatrix} \\ \begin{bmatrix} 15 & 19 & 9 & 1 & 0 & 0 \end{bmatrix} & [1] & [0] \end{array}\right] \tag{29}$$

When $\rho = 16$, $\alpha = 12$ and $\beta = 3$, the central solution for the standard H_∞ problem is given by

$$\begin{aligned} K(s) = & \frac{37688(s+1.0647+j1.7648)}{(s+25.143)(s+11.1196+j14.0465)} \\ \times & \frac{(s+1.0647-j1.7648)(s+5)(s+3)(s+1)}{(s+11.1196-j14.0465)(s+5)(s+3)(s+1)} \end{aligned} \tag{30}$$

Cancel the same stable poles and transmission zeros: -5, -3 and -1 which correspond to the zeros of $M(s)$, we can obtain a 3 order H_∞ controller which is proved by our theorem.

$$\begin{aligned} K(s) &= \frac{37688(s+1.0647+j1.7648)}{(s+25.143)(s+11.1196+j14.0465)} \\ &\times \frac{(s+1.0647-j1.7648)}{(s+11.1196-j14.0465)} \end{aligned} \tag{31}$$

The poles of the designed closed loop are as follows

$$-12.5277 \pm j0.6681, \quad -10.6525,$$
$$-7.7096, \quad -1.9824 \pm j2.9080$$

and the H_∞-norm of the cost function is satisfied since $\|LFT(G,K)\|_\infty = 0.9044 \quad (< 1)$.

The frequency respones of S, W_s^{-1} and T, W_t^{-1} are shown in Figure 2 and Figure 3 by the solid line and the dashed line respectively.

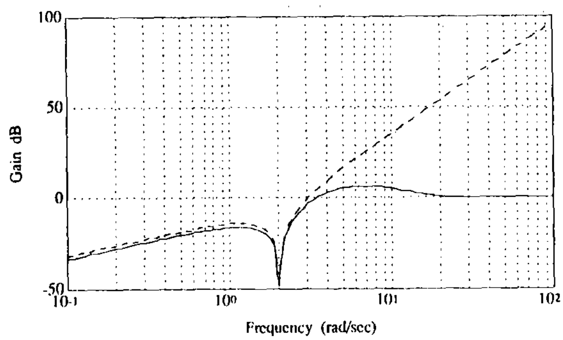

Fig.2. Singular values of $S(s)$ and W_s^{-1}.

4 Conclusions

In this paper, we have presented a mathodology of reducing the order of the H_∞ controllers, for the mixed sensitivity H_∞ design problem in single input single output system, even if the control object and the weighting functions for the sensitivity function have some poles on the imaginary axis. The key point of this methodology is to partition the control object as two parts. One part of them has all the poles of the control object on the imaginary axis; another one has the orther poles. Using this partition, the construction and the state-space representation of the generalized plant $G(s)$ are shown to be simple. So that, this methodology makes it possible to use the standard method to slove the general mixed sensitivity H_∞ design problem and to reduce the order of the designed controllers into n_p.

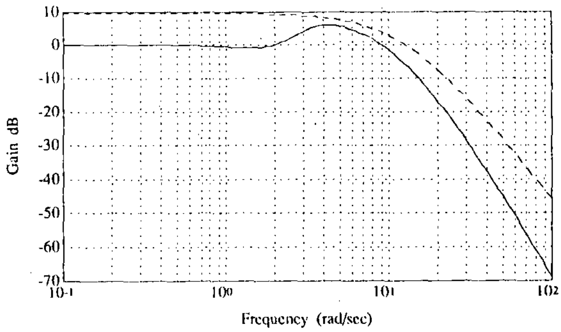

Fig.3. Singular values of $T(s)$ and W_t^{-1}.

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