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Nonlinear Control of A Double-Effect Evaporator by Riemannian Geometric Approach

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Abstract: The purpose of this paper is to present the details of design procedure of a nonlinear regulator by Riemannian geometric approach and to apply it to the case of a double-effect evaporator. A nonlinear geometric model is proposed on a direct sum space of a state vector and a control vector as well as in the previous papers by the authors. The geometric model is derived by replacing the orthogonal straight coordinate axes of a linear system on the direct sum space with the curvilinear coordinate axes. The integral manifold of the geometric model becomes homeomorphic to that of a fictitious linear system. For the geometric model a nonlinear regulator with a performance index is designed renewedly by the procedure of optimization. The construction method of the curvilinear coordinate axes on which the nonlinear system behaves as a linear system is discussed. To apply the above regulator theory to double-effect evaporators especially to the pilot plant at the University of Alberta, a suitable nonlinear model is determined by the plant dynamics. The optimal control law is derived through the calculation of the homeomorphism. As a result it is confirmed that the regulator is effective and superior to that of the conventional control.

1. Introduction

Most of the real plants have some nonlinearities such as constructive nonlinearities, saturation in manipulation and so on. These real plants are usually regarded as linear plants and control systems are designed for the linear plants. However, the nonlinearities can not be ignored to realize the more precise control, especially for the large dynamic range.

The purpose of this paper is to present the details of design procedure of a nonlinear regulator by Riemannian geometric approach and to apply it to the case of the double-effect evaporator which had been studied by Newell and Fisher [1],[2].

The nonlinear regulator by Riemannian geometric approach is constructed based on the following idea: If the trajectories of a linear system is observed on a set of curvilinear coordinate axes instead of the one of the orthogonal straight axes, then the trajectories behave as a nonlinear system. Conversely, the trajectries of a nonlinear system can be treated as those of a linear system by using a suitable curvilinear coordinate axes [3]. In this paper the direct sum of the state vector space and the control vector space is regarded as a Riemannian space. And the nonlinear regulator is derived by replacing the orthogonal straight coordinate axes of the above direct sum space to the curvilinear coordinate axes. Therefore, the integral manifold of this nonlinear regulator is homeomorphic to that of the linear regulator. The basic properties of the linear regulator are to be reflected to the nonlinear regulator.

First, the nonlinear system model is derived in the direct sum space by using the suitable curvilinear coordinate axes.

Next, it is discussed how to distort the curvilinear coordinate axes fitted to the nonlinear system, and get a partial differential equation with respect to the homeomorphism. A nonlinear regulator can be designed with the solution of a Riccati equation for the fictitious linear regulator and has the state feedback form.

The computational algorithm to realize the nonlinear regulator is developed by using the characteristic equation of the partial differential equation.

A nonlinear controller for the double-effect evaporator has been designed, which is a nonlinear system of three inputs and the fifth order. As the result, the performance of the nonlinear control by the Riemannian geometric method is proved to be superior to that of the conventional control.

2. Riemannian geometric model

Consider a nonlinear system

$$\dot{x} = a(x,u)x + b(x,u)u. \tag{1}$$

Let the following system (2) be a fictitious linear system which is paired with the nonlinear system (1).

$$\dot{X} = AX + BU \tag{2}$$

Where x, X are two n-dimensional vectors, and u, U are two r-dimensional vectors, $a(x,u), A$ are $n \times n$ matrixes, $b(x,u), B$ are $n \times r$ matrixes.

Let $\tilde{\mathcal{X}}, \bar{\mathcal{X}}$ be the vectors which span the above direct sum space

$$\tilde{\mathcal{X}} = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \bar{\mathcal{X}} = \begin{pmatrix} X \\ U \end{pmatrix}. \tag{3}$$

Then the equations (1) and (2) are described as

$$\dot{\tilde{\mathcal{X}}} = \tilde{\alpha}(\tilde{\mathcal{X}})\, \tilde{\mathcal{X}} \tag{4}$$

$$\dot{\bar{\mathcal{X}}} = \bar{A}\, \bar{\mathcal{X}}, \tag{5}$$

respectively, where $\tilde{\alpha}(\tilde{\mathcal{X}})$ and $\bar{A}$ are

$$\tilde{\alpha}(\tilde{\mathcal{X}}) = \begin{pmatrix} a(x,u) & b(x,u) \\ 0 & 0 \end{pmatrix} \tag{6}$$

$$\bar{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \tag{7}$$

Now the definition of tensor and its representation by the matrix are given.

Let U and U^* be an n-dimensional vector space and its dual space. An (r,s)-tensor F is defined as a multilinear map

$$F: \quad \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \longmapsto R,$$

where R is a set of numbers in which the addition and the product are defined naturally.

Let U and $\tilde{U}$ be two coordinate neighborhoods on an n-dimensional manifold M with the local coordinate $(x^1, \ldots, x^n)$ and $(\tilde{x}^1, \ldots, \tilde{x}^n)$ respectively. For each x on U, let $T_x(M)$ be a tangent space of M at x, then $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ becomes a basis of $T_x(M)$. If $T_x(M)$ is selected as V^*, then the dual vector space V has a dual basis $(dx^1, \ldots, dx^n)$. Therefore an (r,s)-tensor F is redefined as

$$F = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}} F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}, \qquad (8)$$

where $\otimes$ is a tensor product, and r and s are a contravariant index and a covariant index, respectively. Changing the coordinate system, the components of a tensor F on $U \cap \tilde{U}$ are transferred as

$$\begin{aligned} &F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \qquad (9)\\ &= \sum_{\substack{\mu_1,\ldots,\mu_r \\ \lambda_1,\ldots,\lambda_s}} \frac{\partial x^{i_1}}{\partial \tilde{x}^{\mu_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\mu_r}} \frac{\partial \tilde{x}^{\lambda_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\lambda_s}}{\partial x^{j_s}} \tilde{F}^{\mu_1,\ldots,\mu_r}_{\lambda_1,\ldots,\lambda_s}. \end{aligned}$$

Using the Einstein summation convention, the symbol Σ is usually omitted.

Definition 1: Since the tensor is a multilinear map, an (r,s)-tensor in an n-dimensional space is expressed as an $n^r \times n^s$ matrix whose

$$\begin{aligned} &[i_r + n(i_{r-1} - 1) + n^2(i_{r-2} - 1) + \cdots + n^{r-1}(i_1 - 1),\\ &j_s + n(j_{s-1} - 1) + n^2(j_{s-2} - 1) + \cdots + n^{s-1}(j_1 - 1)] \end{aligned}$$

element is $F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}$.

Using the above definitions, a Riemannian geometric model is derived. Let $\tilde{\mathcal{X}}^\mu$, $\tilde{\mathcal{A}}^\mu_\nu$ be two tensors on the curvilinear coordinate system $(\tilde{x}^i)$, $i = 1, \ldots, n + r$, and $\bar{\mathcal{X}}^\mu, \bar{\mathcal{A}}^\mu_\nu$ be the representations of these tensors on the orthogonal straight coordinate system $(\bar{x}^i)$, respectively. By the transformation formula of a tensor component, we have

$$\bar{\mathcal{X}}^\mu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\gamma} \tilde{\mathcal{X}}^\gamma \qquad (10)$$

$$\bar{\mathcal{A}}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu} \tilde{\mathcal{A}}^\alpha_\beta. \qquad (11)$$

According to the definition of the matrix representation of a tensor, a contravariant vector is expressed as a column vector, and (1,1)-tensor T^μ_ν is expressed as a matrix with (μ, ν) element T^μ_ν. Therefore a linear system (5) is represented as a tensor equation

$$\frac{d}{dt}\bar{\mathcal{X}}^\mu = \bar{\mathcal{A}}^\mu_\nu \bar{\mathcal{X}}^\nu. \qquad (12)$$

Substituting (10) and (11) into (12), we have

$$\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\gamma} \frac{d\tilde{\mathcal{X}}^\gamma}{dt} + \frac{\partial^2 \bar{x}^\mu}{\partial \tilde{x}^\beta \partial \tilde{x}^\lambda} \frac{dx^\lambda}{dt} \tilde{\mathcal{X}}^\beta = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu} \tilde{\mathcal{A}}^\alpha_\beta \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\rho} \tilde{\mathcal{X}}^\rho. \qquad (13)$$

Multiplying $\frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\mu}$ into (13), we have

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} + \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\mu} \frac{\partial^2 \bar{x}^\mu}{\partial \tilde{x}^\beta \partial \tilde{x}^\lambda} \frac{d\tilde{x}^\lambda}{dt} \tilde{\mathcal{X}}^\beta = \tilde{\mathcal{A}}^\gamma_\rho \tilde{\mathcal{X}}^\rho. \qquad (14)$$

Using the Christoffel symbols (14) becomes

$$\widetilde{\{{}_\lambda{}^\mu{}_\nu\}} = \frac{1}{2} g^{\nu k} \left(\frac{\partial g_{\lambda k}}{\partial \tilde{x}^\mu} + \frac{\partial g_{k\mu}}{\partial \tilde{x}^\lambda} - \frac{\partial g_{\mu\lambda}}{\partial \tilde{x}^k} \right) = \frac{\partial \tilde{x}^\nu}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^i}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu}, \qquad (15)$$

then we have

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} + \widetilde{\{{}_\beta{}^\gamma{}_\lambda\}} \frac{d\tilde{x}^\lambda}{dt} \tilde{\mathcal{X}}^\beta = \tilde{\mathcal{A}}^\gamma_\rho \tilde{\mathcal{X}}^\rho \qquad (16)$$

Theorem 1: The linear system (12) descrived on the orthogonal straight coordinate system $(\bar{x}^i), i = 1, \ldots, n + r$, is represented on the curvilinear coordinate system $(\tilde{x}^i)$ as the equation (16).

Proof: Proof is given as above.

Next, we consider the dual model of this Riemannian geometric model (16), The dual model is derived from the transposed linear model

$$\dot{\bar{\mathcal{X}}}^t = \bar{\mathcal{X}}^t \bar{\mathcal{A}}^t \qquad (17)$$

by representing on the curvilinear coordinate system in stead of the orthogonal straight coordinate system.

Using the Riemannian metric tensor $g_{\mu\nu}$, the covariant vector $\tilde{\mathcal{X}}_\nu$ is expressed by means of the contravariant vector $\tilde{\mathcal{X}}^\nu$.

$$\tilde{\mathcal{X}}_\nu = g_{\mu\nu} \tilde{\mathcal{X}}^\nu \qquad (18)$$

From the definition of the matrix representation of a tensor, a covariant vector is expressed as a row vector. Therefore the equation (17) is represented as the tensor equation

$$\frac{d\bar{\mathcal{X}}_\mu}{dt} = \bar{\mathcal{X}}_\nu \overline{{}^t\mathcal{A}}^\nu_\mu, \qquad (19)$$

where ${}^t\mathcal{A}^\nu_\mu$ is a tensor and have the following properties.

$$\overline{{}^t\mathcal{A}}^i_j = \bar{\mathcal{A}}^j_i \qquad (20)$$

Theorem 2: The transposed linear system (19) described on the orthogonal straight coordinate system $(\bar{x}^i), i = 1, \ldots, n + r$, is represented on the curvilinear coordinate system $(\tilde{x}^i)$ as

$$\frac{d\tilde{\mathcal{X}}_\rho}{dt} - \tilde{\mathcal{X}}_\beta \frac{d\tilde{x}^\delta}{dt} \widetilde{\{{}_\delta{}^\beta{}_\rho\}} = \tilde{\mathcal{X}}_\beta \widetilde{{}^t\mathcal{A}}^\beta_\rho. \qquad (21)$$

Proof: By the transformation formula of a tensor component, we have

$$\bar{\mathcal{X}}_\mu = \tilde{\mathcal{X}}_\beta \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\mu}. \qquad (22)$$

Substituting this equation into (19), we have

$$\frac{d\tilde{\mathcal{X}}_\beta}{dt} \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\mu} + \tilde{\mathcal{X}}_\beta \frac{d\bar{x}^\lambda}{dt} \frac{\partial^2 \tilde{x}^\beta}{\partial \bar{x}^\lambda \partial \bar{x}^\mu} = \tilde{\mathcal{X}}_\beta \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu} \overline{{}^t\mathcal{A}}^\nu_\mu. \qquad (23)$$

When the coordinate system is changed, the Christoffel symbols do not behave as tensors and follow the relation

$$\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu} \overline{\{{}_\lambda{}^\nu{}_\mu\}} = \frac{\partial \tilde{x}^\delta}{\partial \bar{x}^\lambda} \frac{\partial \tilde{x}^\rho}{\partial \bar{x}^\mu} \widetilde{\{{}_\delta{}^\beta{}_\rho\}} + \frac{\partial^2 \tilde{x}^\beta}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}. \qquad (24)$$

In the special case, on the orthogonal straight coordinate system the Christoffel symbols become $\overline{\{\lambda{}^{\nu}{}_{\mu}\}} = 0$, therefore we have

$$\frac{\partial^2 \tilde{x}^\beta}{\partial \bar{x}^\lambda \partial \bar{x}^\mu} = -\frac{\partial \tilde{x}^\delta}{\partial \bar{x}^\lambda}\frac{\partial \tilde{x}^\rho}{\partial \bar{x}^\mu}\widetilde{\{\delta{}^{\beta}{}_{\rho}\}} \tag{25}$$

Multiplying $\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho}$ into the equation (23) on the right side, we have

$$\frac{d\tilde{X}_\rho}{dt} + \tilde{X}_\beta \frac{d\bar{x}^\lambda}{dt}\frac{\partial^2 \tilde{x}^\beta}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho} = \tilde{X}_\beta \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\overline{{}^t\mathcal{A}}^\nu{}_\mu \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho}. \tag{26}$$

Using (25), we have the equation (21) Q.E.D.

3. Homeomorphism

In this section, we consider the mapping between two integral manifolds of the Riemannian geometric model on the orthogonal straight coordinate system and the curvilinear coordinate system. Let $\bar{U}$ and $\tilde{U}$ be the coordinate neighborhoods on these integral manifolds, respectively. Let τ^ν_γ, T^γ_ν be

$$\tau^\nu_\gamma = \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\gamma}, \quad T^\gamma_\nu = \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\nu}.$$

Then we can consider the tranformation formula of a tensor component

$$\bar{X}^\nu = \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\gamma}\tilde{X}^\gamma, \tag{27}$$

$$\bar{X}_\nu = \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\nu}\tilde{X}_\gamma \tag{28}$$

as to be the mappings between $\bar{X}^\nu \in \bar{U}$ and $\tilde{X}^\gamma \in \tilde{U}$ with respect to τ and T. If τ and T are both continuous mappings, then the mapping τ becomes the homeomorphism between two integral manifolds.

Substituting these τ and T into (14) and (26), the Riemannian geometric model (16) and its dual model (21) become

$$\frac{d\tilde{X}^\gamma}{dt} = (T^\gamma_\mu \bar{A}^\mu{}_\lambda \tau^\lambda_\rho - T^\gamma_\mu \frac{d\tau^\mu_\rho}{dt})\,\tilde{X}^\rho \tag{29}$$

$$\frac{d\tilde{X}_\rho}{dt} = \tilde{X}_\beta(T^\beta_\nu \overline{{}^t\mathcal{A}}^\nu{}_\mu \tau^\mu_\rho - \frac{dT^\beta_\mu}{dt}\tau^\mu_\rho). \tag{30}$$

4. Nonlinear optimal regulator

Theorem 3: When the homeomorphism τ exists and is represented as

$$(\tau^i_j) = \begin{matrix} & n & r \\ n & \tau_{11} & \tau_{12} \\ r & \tau_{21} & \tau_{22} \end{matrix}, \tag{31}$$

then for the Riemannain geometric model (16) and the performance index

$$J = \frac{1}{2}\int_{t_0}^{T} \tilde{X}_i T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_j \tilde{X}^j \; dt \tag{32}$$

$$(\bar{Q}^\nu{}_\mu) = \begin{matrix} & n & r \\ n & Q & 0 \\ r & 0 & R \end{matrix}, \tag{33}$$

we have the optimal control law

$$u = -(\tau_{22} + K\tau_{12})^{-1}(\tau_{21} + K\tau_{11})x \tag{34}$$

$$K = R^{-1}B^tS. \tag{35}$$

Where S is the solution of the Riccati equation

$$\frac{dS}{dt} + SA + A^tS - SBR^{-1}B^tS + Q = 0. \tag{36}$$

$$S(T) = 0 \tag{37}$$

Proof: Since the Hamiltonian function H is a scalar function and is invariant to the transformation of the coordinate system, the calculus of variation to the control problem can be applicable to our Riemannian geometric approach. The proof is given by the procedure of optimization in [4]. The Hamiltonian H for the Riemannian geometric model (16) with the cost J of (32) is

$$H = \frac{1}{2}(\tilde{X}_i T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_j \tilde{X}^j) + \psi_j(T^j_\mu \bar{A}^\mu{}_\lambda \tau^\lambda_\rho - T^j_\mu \frac{d\tau^\mu_\rho}{dt})\tilde{X}^\rho, \tag{38}$$

where ψ_j is a costate covariant vector. The canonical equation is derived as

$$\frac{d\psi_\alpha}{dt} = -\frac{\partial H}{\partial \tilde{X}^\alpha} = -g_{i\alpha}\frac{\partial H}{\partial \tilde{X}_i}$$

$$= -\frac{1}{2}g_{i\alpha}T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_j \tilde{X}^j - \frac{1}{2}\tilde{X}_i T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_\alpha - \psi_j[T^j_\mu \bar{A}^\mu{}_\lambda \tau^\lambda_\alpha - T^j_\mu \frac{d\tau^\mu_\alpha}{dt}], \tag{39}$$

with the boundary condition

$$\psi_\alpha(T) = 0. \tag{40}$$

Supose that

$$\psi_\alpha = \tilde{X}_i \tilde{S}^i_\alpha = \tilde{X}_i T^i_\nu \bar{S}^\nu_\mu \tau^\mu_\alpha \tag{41}$$

$$(\bar{S}^\nu_\mu) = \begin{matrix} & n & r \\ n & S & 0 \\ r & 0 & 0 \end{matrix}, \tag{42}$$

then from (39) we have

$$\frac{d\psi_\alpha}{dt} = -\frac{1}{2}g_{i\alpha}T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_j \tilde{X}^j - \frac{1}{2}\tilde{X}_i T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_\alpha - \tilde{X}_i T^i_\nu \bar{S}^\nu_\mu \bar{A}^\mu{}_\lambda \tau^\lambda_\alpha + \tilde{X}_i T^i_\nu \bar{S}^\nu_\mu \frac{d\tau^\mu_\alpha}{dt}. \tag{43}$$

On the other hand, by differentiating (41) with t, and using the dual model (30), we have

$$\frac{d\psi_\alpha}{dt} = \tilde{X}_j\{T^j_m \overline{{}^t\mathcal{A}}^m{}_n \tau^n_i - \frac{dT^j_n}{dt}\tau^n_i\}T^i_\nu \bar{S}^\nu{}_\mu \tau^\mu_\alpha$$

$$+ \tilde{X}_j \frac{dT^j_\nu}{dt}\bar{S}^\nu_\mu \tau^\mu_\alpha + \tilde{X}_j T^j_\nu \frac{d\bar{S}^\nu_\mu}{dt}\tau^\mu_\alpha + \tilde{X}_j T^j_\nu \bar{S}^\nu_\mu \frac{d\tau^\mu_\alpha}{dt}$$

$$= \tilde{X}_j T^j_m \overline{{}^t\mathcal{A}}^m{}_n \bar{S}^n_\mu \tau^\mu_\alpha + \tilde{X}_j T^j_\nu \frac{d\bar{S}^\nu_\mu}{dt}\tau^\mu_\alpha + \tilde{X}_j T^j_\nu \bar{S}^\nu_\mu \frac{d\tau^\mu_\alpha}{dt}. \tag{44}$$

Comparing (43) with (44), we have

$$0 = \frac{1}{2} g_{i\alpha} T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_j \tilde{\mathcal{X}}^j \quad + \tilde{\mathcal{X}}_j T^j_m [\frac{1}{2}\bar{Q}^m_\mu + \bar{\mathcal{S}}^m_n \bar{\mathcal{A}}^n_\mu + \overline{{}^t\mathcal{A}}^m_n \bar{\mathcal{S}}^n_\mu + \frac{d\bar{\mathcal{S}}^m_\mu}{dt}]\tau^\mu_\alpha$$

$$= \frac{1}{2} g_{i\alpha} T^i_\nu \bar{Q}^\nu_\mu \tau^\mu_j \tilde{\mathcal{X}}^j + \tilde{\mathcal{X}}^\alpha g_{j\alpha} T^j_m [\frac{1}{2}\bar{Q}^m_\mu + \bar{\mathcal{S}}^m_n \bar{\mathcal{A}}^n_\mu + \overline{{}^t\mathcal{A}}^m_n \bar{\mathcal{S}}^n_\mu + \frac{d\bar{\mathcal{S}}^m_\mu}{dt}]\tau^\mu_\alpha$$

$$= g_{j\alpha} T^j_m [\bar{Q}^m_\mu + \bar{\mathcal{S}}^m_n \bar{\mathcal{A}}^n_\mu + \overline{{}^t\mathcal{A}}^m_n \bar{\mathcal{S}}^n_\mu + \frac{d\bar{\mathcal{S}}^m_\mu}{dt}]\tau^\mu_\alpha \tilde{\mathcal{X}}^\alpha. \qquad (45)$$

Since this equation always holds good, and $g_{j\alpha}T^j_m$ is not always 0, we have

$$(\frac{d\bar{\mathcal{S}}^m_\mu}{dt} + \bar{\mathcal{S}}^m_n \bar{\mathcal{A}}^n_\mu + \overline{{}^t\mathcal{A}}^m_n \bar{\mathcal{S}}^n_\mu + \bar{Q}^m_\mu)\tau^\mu_\alpha \tilde{\mathcal{X}}^\alpha = 0. \qquad (46)$$

Representing (46) as a matrix form with (3), (7), (31), (33) and (42), we have the equations (34) and (36). Furthermore, by using the relation (41), the boundary condition (40) becomes

$$\bar{\mathcal{S}}^m_\mu(T) = 0 \qquad (47)$$

Q.E.D.

5. Gauge field

Representing the nonlinear system (4) with the tensors, we have

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} = \tilde{\alpha}^\gamma_\mu(\tilde{\mathcal{X}})\, \tilde{\mathcal{X}}^\mu. \qquad (48)$$

In this section the curvilinear coordinate system is constructed on which the nonlinear system (48) is obseved as a linear system (12). Since the equation (48) is equivalent to the Riemannian geometric model (16), we have

$$\bar{\mathcal{A}}^\gamma_\mu - \widetilde{\{{}_\mu{}^\gamma{}_\lambda\}}\frac{d\tilde{x}^\lambda}{dt} = \tilde{\alpha}^\gamma_\mu(\tilde{\mathcal{X}}) \qquad (49)$$

Using (11) and the relations

$$\widetilde{\{{}_\mu{}^\gamma{}_\lambda\}} = T^\gamma_k \frac{\partial \tau^k_\mu}{\partial \tilde{x}^\lambda}, \qquad \frac{\partial \tau^k_\mu}{\partial \tilde{x}^\lambda}\frac{d\tilde{x}^\lambda}{dt} = \frac{d\tau^\beta_\mu}{dt} = \frac{\partial \tau^\beta_\mu}{\partial \tilde{\mathcal{X}}^\lambda}\frac{d\tilde{\mathcal{X}}^\lambda}{dt},$$

the equation (49) becomes the following partial differential equation

$$\frac{\partial \tau^\beta_\mu}{\partial \tilde{\mathcal{X}}^\lambda}[\tilde{\alpha}^\lambda_\gamma(\tilde{\mathcal{X}})\, \tilde{\mathcal{X}}^\gamma] = \bar{\mathcal{A}}^\beta_\nu \tau^\nu_\mu - \tau^\beta_\gamma\, \tilde{\alpha}^\gamma_\mu(\tilde{\mathcal{X}}). \qquad (50)$$

Theorem 4: The homeomorphism τ between the integral manifold of a nonlinear system (4) and that of a linear system (5) satisfies a partial differential equation (50).

Since this equation is a quasi-linear partial differential equation of first order, we have the characteristic equations.

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} = \tilde{\alpha}^\gamma_\mu(\tilde{\mathcal{X}})\, \tilde{\mathcal{X}}^\mu \qquad (51)$$

$$\frac{d\tau^\beta_\mu}{dt} = \bar{\mathcal{A}}^\beta_\nu \tau^\nu_\mu - \tau^\beta_\gamma\, \tilde{\alpha}^\gamma_\mu(\tilde{\mathcal{X}}). \qquad (52)$$

Proof: Proof is given as above.

6. Double-effect evaporator

The flow diagram of the double effect evaporator is shown in Fig.1. In this paper the plant parameters are adopted from that of the pilot plant at the University of Alberta. The process variables and their steady state values are shown in Table 1 [1],[2].

This plant consists of two evaporators which are connected sequentially. The material flows into the first step evaporator with the flow rate F, and heated by the steam. The product of the first step evaporator flows into the second step evaporator with the flow rate B1 and heated by the overhead vapor of the first step evaporator. The product of the second step evaporator is taken out with the flow rate B2. The overhead vapor of the second step evaporator is condensed by the water and taken out as the second product. The vapor which has not been condensed in the second step evaporator is discharged.

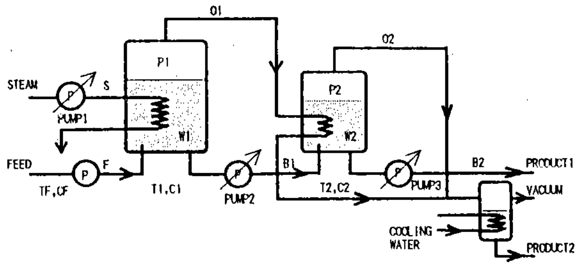


Fig.1: Schematic diagram of a double-effect evaporator

The dynamics of this plant is given in [1].

$$\frac{dW1}{dt} = F - B1 - O1 \qquad (53)$$

$$W1\frac{dC1}{dt} = F(CF - C1) + O1 \cdot C1 \qquad (54)$$

$$W1\frac{dH1}{dt} = F(HF - H1) - O1(HO1 - H1) + Q1 - L1 \qquad (55)$$

$$\frac{dW2}{dt} = B1 - B2 - O2 \qquad (56)$$

$$W2\frac{dC2}{dt} = B1(C1 - C2) + O2 \cdot C2 \qquad (57)$$

$$Q1 = U1 \cdot A1(TS - T1) = \lambda \cdot S \qquad (58)$$

$$Q2 = U2 \cdot A2(T1 - T2) \qquad (59)$$

$$O1 = (Q2 + L2)/(HO1 - HC) \qquad (60)$$

$$O2 = \frac{Q2 - L3 + B1(H1 - H2) + DH2 \cdot B1(C2 - C1)}{HO2 - H2 + DH2 \cdot C2} \qquad (61)$$

$$T1 = (H1 + 32.1)/(1.0 - 0.16 \cdot C1) \qquad (62)$$

$$TS = S(HS + 32.1) + T1 \cdot U1 \cdot A1/(U1 \cdot A1 + S) \qquad (63)$$

$$T2 = (H2 + 32.1)/(1.0 - 0.16 \cdot C2) \qquad (64)$$

$$H2 = T2 \cdot (1.0 - 0.16 \cdot C2) - 32.1 \qquad (65)$$

$$HO1 = 1066.1 + 0.4 \cdot T1 \qquad (66)$$

$$HO2 = 1066.1 + 0.4 \cdot T2 \qquad (67)$$

$$HC = T1 - 32.1 \qquad (68)$$

$$DH2 = \frac{\partial\, H2}{\partial\, C2} \qquad (69)$$

In [1], the plant is treated as a 3 inputs 5th order controlled system (5NL MODEL) and the controller is designed as a linear controlled system. In this paper, the plant is designed as a 3 inputs 5th order nonlinear system with the form of (1). The system variables are transformed as

$$
\begin{aligned}
x_1 &= W1 - \overline{W1} && (70)\\
x_2 &= C1 - \overline{C1} && (71)\\
x_3 &= H1 - \overline{H1} && (72)\\
x_4 &= W2 - \overline{W2} && (73)\\
x_5 &= C2 - \overline{C2} && (74)\\
u_1 &= S - \overline{S} && (75)\\
u_2 &= B1 - \overline{B1} && (76)\\
u_3 &= B2 - \overline{B2}, && (77)
\end{aligned}
$$

State variables		
W1	liquid holdup in the first effect	30 lb
C1	product concentration from the first effect	0.0485
H1	liquid enthalpy in the first effect	194 Btu/lb
W2	liquid holdup in the second effect	35 lb
C2	product concentration from the second effect	0.0965
Control variables		
S	steam flowrate into the first effect	1.9 lb/min
B1	bottoms flowrate from the first effect	3.3 lb/min
B2	bottoms flowrate from the second effect	1.659 lb/min
Load variables		
F	feed flowrate	5.0 lb/min
CF	feed concentration	0.032
HF	feed enthalpy	162 Btu/lb
Other variables		
O1	overhead vapor from the first effect	1.701 lb/min
O2	overhead vapor from the second effect	1.641 lb/min
P1	pressure in the first effect	< 25.0 psia
P2	pressure in the second effect	7.5 psia
TF	temperature of feed	190 °F
T1	temperature in the first effect	227.9 °F
T2	temperature in the second effect	141.5 °F
TS	temperature of steam	255.0 °F
HO1	first effect overhead vapor enthalpy	1157.3 Btu/lb
HO2	second effect overhead vapor enthalpy	1122.7 Btu/lb
Q1	heat flow rate in the first effect	1831.1 Btu/min
Q2	heat flow rate in the second effect	1610.8 Btu/min
H2	liquid enthalpy in the second effect	107.2 Btu/lb
H3	steam enthalpy	1184.7 Btu/lb
HC	liquid enthalpy in the condenser	Btu/lb
A	heat transfer area	ft ^{2}
U	heat transfer coefficient	Btu/min/°F/ft ^{2}
U1 · A1		67.567 Btu/min/°F
U2 · A2		18.644 Btu/min/°F
L1	heat loss	31.943 Btu/min
L2	heat loss	25.0 Btu/min
L3	heat loss	230.21 Btu/min
DH2		-22.64 Btu/lb
λ		963.7 Btu/lb

Table.1: Steady values of process variables

where the sign $\overline{}$ means a steady state value. Now, in this paper, the vapors $O1$ and $O2$ are not treated as the state variables. Therefore by the relations (58)~(69), $O1, O2, HO1$ are approximated as

$$
\begin{aligned}
O1 &= \overline{\alpha}_1\, x_2 + \overline{\alpha}_2\, x_3 + \overline{O1} && (78)\\
O2 &= \overline{\beta}_1\, x_2 + \overline{\beta}_2\, x_3 + \overline{\beta}_3\, x_5 + \overline{\beta}_4\, u_2 + \overline{O2} && (79)\\
HO1 &= \overline{\gamma}_1\, x_2 + \overline{\gamma}_2\, x_3 + \overline{HO1} && (80)\\
\overline{\alpha}_1 &= 2.74,\ \overline{\alpha}_2 = 0.195,\ \overline{\beta}_1 = 0.494,\ \overline{\beta}_2 = 0.0218\\
\overline{\beta}_3 &= -0.0737,\ \overline{\beta}_4 = 0.0846,\ \overline{\gamma}_1 = -171.9,\ \overline{\gamma}_2 = 0.403.
\end{aligned}
$$

The state equations are represented as

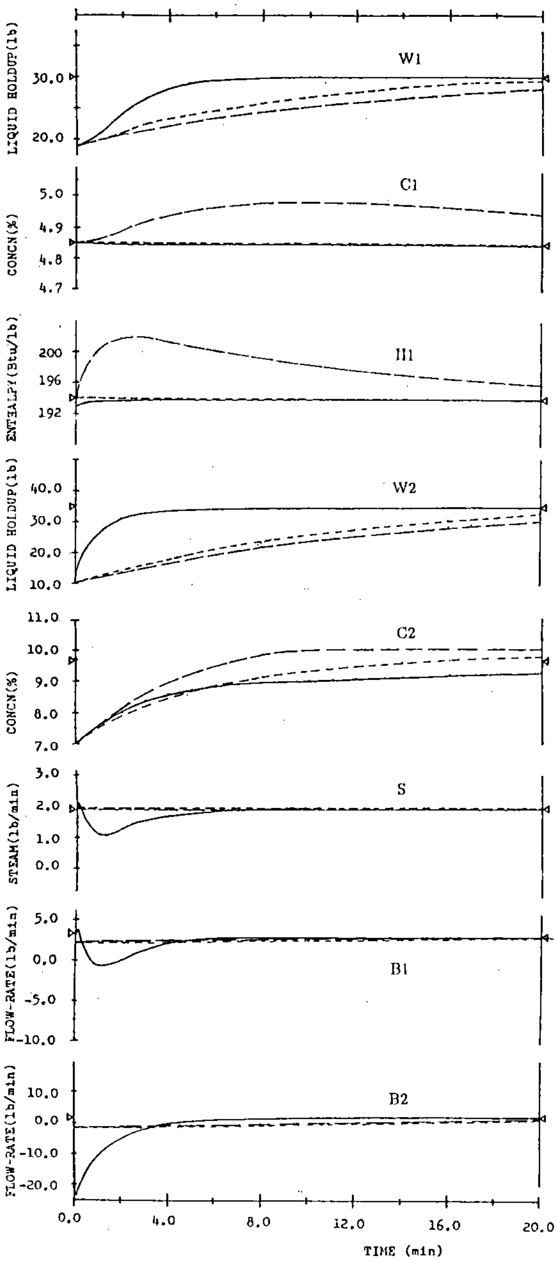
$$
\begin{aligned}
\frac{dx_1}{dt} &= -\overline{\alpha}_1 x_2 - \overline{\alpha}_2 x_3 - u_2 && (81)\\
\frac{dx_2}{dt} &= \frac{-F + (x_2 + \overline{C1})\overline{\alpha_1} + \overline{O1}}{x_1 + \overline{W1}} x_2 + \frac{(x_2 + \overline{C1})\overline{\alpha}_2}{x_1 + \overline{W1}} x_3 && (82)\\
\frac{dx_3}{dt} &= \frac{-(\overline{\gamma}_1 x_2 + \overline{\gamma}_2 x_3 + \overline{HO1})\overline{\alpha}_1 - \overline{\gamma}_1 \overline{O1}}{x_1 + \overline{W1}} x_2\\
&+ \frac{(\overline{\gamma}_1 x_2 + \overline{HO1} x_3 + \overline{HO1})\overline{\alpha}_2 - \overline{\gamma}_2 \overline{O1} - \overline{B1}}{x_1 + \overline{W1}} x_3\\
&+ \frac{\lambda}{x_1 + \overline{W1}} u_1 - \frac{x_3 + \overline{H1}}{x_1 + \overline{W1}} u_2 && (83)\\
\frac{dx_4}{dt} &= -\overline{\beta}_1 x_2 - \overline{\beta}_2 x_3 - \overline{\beta}_3 x_5 + (1 - \overline{\beta}_4) u_2 - u_3 && (84)\\
\frac{dx_5}{dt} &= \frac{\overline{B1} + (x_5 + \overline{C2})\overline{\beta}_1}{x_4 + \overline{W2}} x_2\\
&+ \frac{(x_5 + \overline{C2})\overline{\beta}_2}{x_4 + \overline{W2}} x_3 + \frac{-\overline{B1} + (x_5 + \overline{C2})\overline{\beta}_3 + \overline{\beta}_5}{x_4 + \overline{W2}} x_5\\
&+ \frac{(x_2 - x_5 + \overline{C1} - \overline{C2}) + (x_5 + \overline{C2})\overline{\beta}_4}{x_4 + \overline{W2}} u_2. && (85)
\end{aligned}
$$

7. Algorithm and simulation results

The optimal control law in Theorem 3 is constructed from the homeomorphism and the solution of the Riccati equation. When the homeomorphism is derived by the solution of the partial differential equation in Theorem 4, it is important where the integral manifold of the nonlinear model is in contact with that of the linear model. In this paper the initial point is regarded as the contact point. The homeomorphism is calculated by using the characteristic equation (52) along the trajectory of the nonlinear system (51) instead of the direct calculation of the partial differential equation. The simulation results to the double-effect evaporator are shown in Fig.2 and Fig.3. Where the matrices Q and R are taken as unit matrices I_5 and I_3, respectively. As a result, the responce speed of the nonlinear regulator is faster than that of the conventional control.

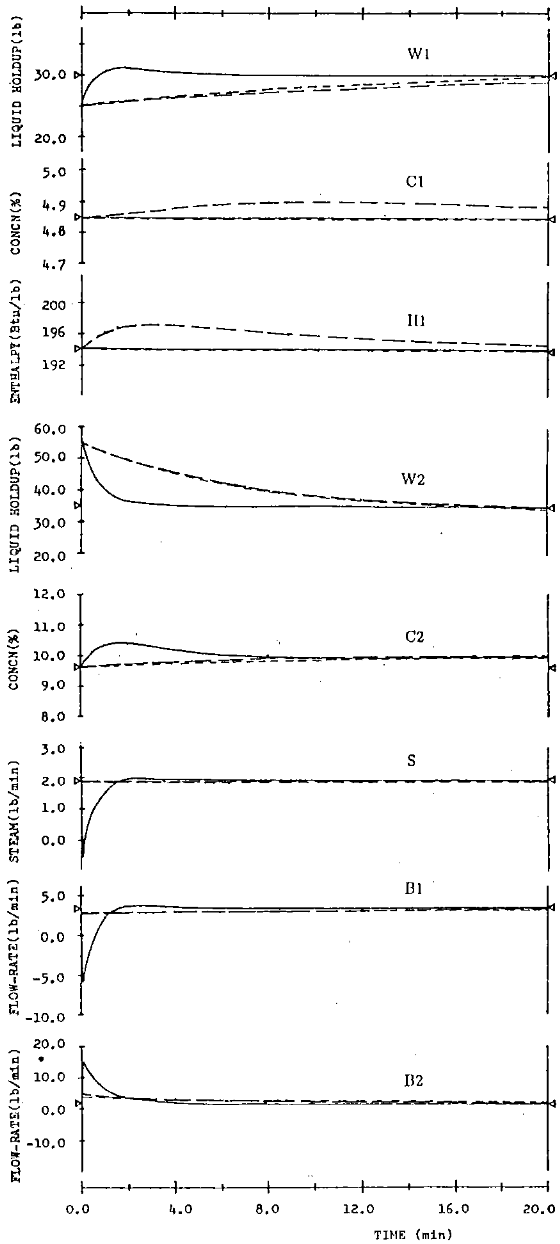
8. Conclusions

By using a suitable curvilinear coordinate axes, the Riemannian geometric model and its dual model have been derived in Theorem 1 and Theorem 2. Because the Riemmanian geometric model is homeomorphic to a linear model, the nonlinear optimal regulator has been derived in Theorem 3 by using this homeomorphism. Furthermore, it has been discussed how to distort the curvilinear coordinate axes fitted to the nonlinear system, and a partial differential equation with respect to the homeomorphism has been derived in Theorem 4. Applying this nonlinear regulator theory to the double-effect evaporator, the usefulness of this regulator has been confirmed.



- - - - 5NL MODEL (PI-CONTROL),
—— —— $\dot{x} = a(x,u)x + b(x,u)u$ (PI-CONTROL),
———— $\dot{x} = a(x,u)x + b(x,u)u$ (NL-REGURATOR)

Fig.2: Trajectories of controled evaporator (case 1)



- - - - 5NL MODEL (PI-CONTROL),
—— —— $\dot{x} = a(x,u)x + b(x,u)u$ (PI-CONTROL),
———— $\dot{x} = a(x,u)x + b(x,u)u$ (NL-REGURATOR)

Fig.3: Trajectories of controled evaporator (case 2)

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