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Stability Analysis to Generally Structured Robust Control Problems

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Abstract

In this paper, the robust stability of characteristic polynomials with respect to real parameter variations is investigated through a new functional approach. Specifically there is no restriction on the interrelationship between coefficients of the polynomials. This allows one to treat the robust stability problems alike without distinction as to continuous or discrete time systems. Necessary condition and sufficient condition for the robust stability are shown and some examples extracted from two-link planar manipulator are provided.

Keywords: robust stability, holomorphic, maximum modulus principle, meromorphic, Rouche's theorem

1 Introduction

Since Kharitonov [1] suggested the seminal stability theorem for an interval family of polynomials, the research on the so-called robust stability problem has been one of the most fascinating areas. Among the plethora of papers on robust stability, Kharitonov's theorem has been extended into more general cases. Kharitonov's assumption that the set of possible coefficient variations is an $(n+1)$-dimensional — n is the order of the characteristic polynomial — rectangle severely inhibits application of the results to practical problems. This rectangularity assumption is tantamount to having independent coefficient perturbations. Moreover the assumption that the predefined region D is the left-half plane makes it impossible apply to discrete time case.

Nowadays more general robust stability criteria are obtained and the amount of calculation is reduced for the practical purpose through the edge theorem [2] and the zero exclusion principle [3]. However all those attempts are ceased at graphical tests, numerical tests or algebraic conditions mainly focusing on characteristic polynomial families which are polytopic [4] — the convex hull of a finite set of points — in a coefficient space. No clear and simple algebraic stability criterion exists for the characteristic polynomial sets with coefficients of multi-linear or nonlinear dependence until now.

In this paper, new necessary conditions and sufficient conditions for the robust stability of discrete time systems are derived by considering the characteristic polynomial itself as an element in the functional space and using an extension of the maximum modulus principle [5] and Rouche's theorem [6]. In derivation no critical interrelationship between coefficients such as linear dependence in polynomial polytope is assumed so the robust stability problem of continuous time systems can be easily transferred to that of discrete time systems through the bilinear transformation($s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$) and it can be analyzed accordingly.

2 Problem formulation

For a given characteristic polynomial with parameter variations of discrete time systems, let it be formulated as

$$f(z) = \sum_{i=0}^{n} a_i z^i \tag{1}$$

with $a_i \in [\overline{a_i} - a_i^*, \overline{a_i} + a_i^*]$ where $\overline{a_i}$ is the nominal value of a_i and a_i^* is the interval variation of a_i. The robust stability problem is now reduced to find certain criterion under which $f(z)$ has all its roots in the unit circle.

3 Robust Stability Criteria

3.1 Necessary Condition

The zeros of an analytic function in a certain region have the effect of making the function small throughout the region. The quantitative results showing how small in terms of the number of zeros were given by Jensen and et al. [5]. We extend the results to solve the robust stability problem.

Theorem 1 (necessary condition) : For the system described by the characteristic polynomial $f(z)$ in (1) to be stable, it should satisfy

$$|a_n| \leq |\sum_{i=0}^{n} a_i(cos\omega i + jsin\omega i)|, \quad \forall a_i \in [\overline{a_i} - a_i^*, \overline{a_i} + a_i^*], \ i = 0, 1, \ldots, n \text{ and } \forall \omega \in [0, 2\pi)$$

Before proving the theorem 1, some notions should be predefined.

Definition 1 : Let f be a function defined in some neighborhood of a point z_0. We say that f is ***holomorphic*** at z_0 if there exists a power series

$$\sum_{k=0}^{\infty} a_k(z - z_0)^k$$

and some $r > 0$ such that the series converges absolutely for $|z - z_0| < r$, and such that for such z, we have

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

Suppose f is a function on an open set U. We say that f is holomorphic on U if f is holomorphic at every point of U.

If S is an arbitrary set, not necessarily open, it is useful to make the convention that a function is holomorphic on S if it is the restriction of a holomorphic function on an open set containing S. This is useful, for instance, when S is a closed disc.

Lemma 1(global version of maximum modulus principle [5]) : Let U be a connected open set, and let f be a holomorphic function on U. If $z_0 \in U$ is a maximum point for f, that is $|f(z_0)| \geq |f(z)|$ for all $z \in U$, then f is constant on U.

Proof of Theorem 1 : $f(z)$ in (1) is holomorphic [5] — that is, power series expansion is possible — on the closed unit disc. Let the zeros of $f(z)$ in the open unit disc be ordered by increasing absolute value with each repeated for its multiplicity respectively, i.e., $|z_1| \leq |z_2| \leq \ldots \leq |z_N|$ where $N \leq n$. Let $\overline{z_m}$ be the complex conjugate of z_m. Then the function

$$g(z) = f(z) \prod_{m=0}^{N} \frac{1 - z\bar{z}_m}{z - z_m}$$

is also holomorphic on the closed unit disc and

$$|g(z)| = |f(z)|$$

on the unit circle $\gamma_u = \{z : |z| = 1\}$. Hence the maximum modulus principle [5] implies that

$$|g(z)| \leq ||f||_{\gamma_u}$$

With $z = 0$, (1) becomes

$$\begin{aligned} |f(0)| &= |a_0| \\ &\leq ||f||_{\gamma_u}|z_1 z_2 \cdots z_N| \\ &= |\sum_{i=0}^{n} a_i(cos\omega i + jsin\omega i)||\frac{a_0}{a_n}|, \quad \forall \omega \in [0, 2\pi) \end{aligned}$$

$$\text{Therefore } |a_n| \leq |\sum_{i=0}^{n} a_i(cos\omega i + jsin\omega i)|, \quad \forall a_i \in [\overline{a_i} - a_i^*, \overline{a_i} + a_i^*], \ i = 0, 1, \ldots, n \text{ and } \forall \omega \in [0, 2\pi)$$

Q.E.D.

Corollary 1 : Especially for the characteristic polynomial $f(z)$ with $\forall a_i \geq 0$, $i = 0, 1, \ldots, n$ in (1), the condition in Theorem 1 reduces to the following.

$$|a_n| \leq |\sum_{i=0}^{n} a_i(cos\frac{2\pi}{n+1}i + jsin\frac{2\pi}{n+1}i)|, \quad \forall a_i \in [\overline{a_i} - a_i^*, \overline{a_i} + a_i^*], \ i = 0, 1, \ldots, n$$

Proof : In the proof of theorem 1, if all the coefficients a_i is positive, then from the viewpoint of symmetric cancellation for the weighted sum of vectors — *that is, the minimum occurs when the vectors arrange in symmetric form such that they are cancelled with each other in vector summation* — , the last condition reduces to the above one.

Q.E.D.

3.2 Sufficient Condition

For the system satisfying the necessary condition in Theorem 1, we can test its sufficiency by the following Theorem 2.

Theorem 2 (sufficient condition) : Suppose that a characteristic polynomial $f(z)$ in (1) is never zero on the unit circle and there exists $g(z) = \sum_{i=0}^{n} b_i z^i, b_i \in [\overline{b_i} - b_i^*, \overline{b_i} + b_i^*]$ which has all its roots inside the unit circle such that $\overline{b_i} = -\overline{a_i}$ and

$b_i^* = a_i^*$ for any $i \in [1, n]$, then the system described by the characteristic polynomial $f(z)$ is stable.

For the proof of the theorem 2, some preliminaries are required.

Definition 2 : If G is open and f is a function defined and analytic in G except for poles, then f is a *meromorphic* function on G.

Lemma 2(Rouche's Theorem [6]*) :* Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a; R)$ with no zeros or poles on the circle $\gamma = \{z : |z - a| = R\}$. If $Z_f, Z_g(P_f, P_g)$ are the number of zeros(poles) of f and g inside γ counted according to their multiplicities and if

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$Z_f - P_f = Z_g - P_g.$$

Proof of Theorem 2 : $f(z)$, $g(z)$ are polynomials so they are meromorphic [6] — that is, analytic — on any domain G. Assume $f(z)$ has no zeros on the unit circle $\gamma_u = \{z : |z| = 1\}$. Then on γ_u,

$$\begin{aligned} |f(z)+g(z)| &= |\sum_{i=0}^{n} a_i z^i + \sum_{i=0}^{n} b_i z^i| \\ &= |\sum_{i=0}^{n} a_i cos\omega i + \sum_{i=0}^{n} b_i cos\omega i + j(\sum_{i=0}^{n} a_i sin\omega i + \\ &\quad \sum_{i=0}^{n} b_i sin\omega i)|, \quad \forall\omega \in [0, 2\pi) \end{aligned}$$

If $a_k = -b_k$, then

$$\begin{aligned} |f(z)+g(z)| &= | \sum_{i=0, i\neq k}^{n} (a_i + b_i) cos\omega i + \sum_{i=0, i\neq k}^{n} (a_i + b_i) sin\omega i| \\ &< |\sum_{i=0}^{n} a_i cos\omega i + j \sum_{i=0}^{n} a_i sin\omega i| + |\sum_{i=0}^{n} b_i cos\omega i + \\ &\quad j \sum_{i=0}^{n} b_i sin\omega i|, \quad \forall\omega \in [0, 2\pi) \\ &= |f(z)| + |g(z)| \end{aligned}$$

Hence the proof follows from Lemma 2.

Q.E.D.

4 Examples

The characteristic polynomial of a two-link planar manipulator [7] is of degree 3 and uncertainty in the physical parameters of the manipulator can result in linearly or nonlinearly dependent coefficient perturbations according to the degree of freedom and mechanical property of each joint.

Example 1 : Let $f(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$ with $a_3 \in [2, 6]$, $a_2 = a_0^2$, $a_1 = 1.5 - a_3$, $a_0 \in [-2, 2]$, then we can find that it does not satisfy the condition in Theorem 1 — that is, it does not satisfy the necessary condition for stability — since for $\omega = \frac{\pi}{16}$, $|a_3| \geq 2 > 1.8714 = |\sum_{i=0}^{3} a_i(cos\omega i + jsin\omega i)|$ with $a_0 = -0.4, a_1 = -2.4, a_2 = 0.16, a_3 = 3.9$. Thus $f(z)$ has some zeros outside the unit circle and the system described by this characteristic polynomial is unstable. We examine it through the root locus of $f(z) = 0$ in Fig. 1.

Example 2 : Let $f(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$ with $a_3 \in [0.5, 1.5]$, $a_2 = a_0^2$, $a_1 = 0.5 - a_3$, $a_0 \in [-0.1, -0.001]$, then we can find that it satisfies the condition in Theorem 1 — that is, it satisfies the necessary condition for stability — and that there exists $g(z)$ such that for $\overline{a_3} = 1$, $\overline{a_0} = -0.0505$ it has the form of $\overline{b_3} = -\overline{a_3}$, $\overline{b_0} = \overline{a_0}$, i.e., the nominal polynomial $\overline{g}(z) = -z^3 - 0.0505$ and b_3, b_0 have the same intervals as a_3, a_0. Let the zeros of $g(z)$ be z_i, $i = 1, 2, 3$, then we can easily find that $|z_i| \leq \frac{1}{\sqrt[3]{5}} < 1$, $\forall i = 1, 2, 3$. Thus $f(z)$ has all its zeros in the unit circle and the system described by this characteristic polynomial also is stable. We also examine it through the root locus of $f(z) = 0$ in Fig. 2.

5 Conclusions

The robust stability of characteristic polynomials with respect to coefficient variations was investigated through a new functional approach without any assumptions on the interrelationship between coefficients. New algebraic conditions for robust stability are obtained for discrete time systems. It is shown that these conditions are applicable to characteristic polynomials with multi-linearly or nonlinearly coupled coefficients. It is also shown that these conditions are available to continuous time systems.

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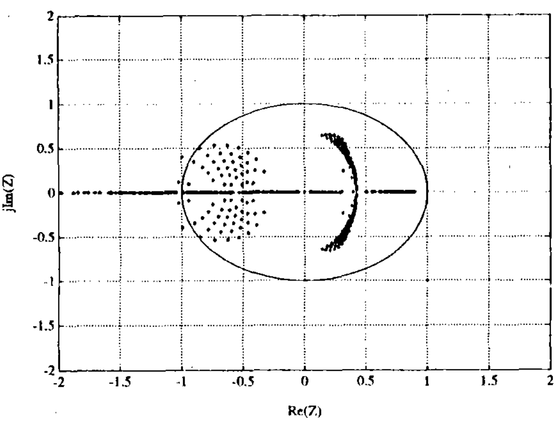

Fig. 1 Root locus of $f(z) = 0$ in Example 1

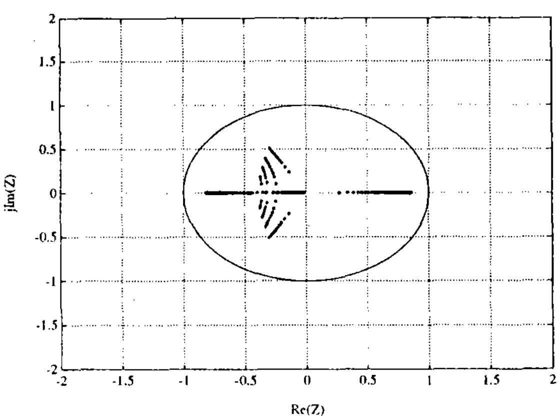

Fig. 2 Root locus of $f(z) = 0$ in Example 2