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Stabilizing Solutions of Algebraic Matrix Riccati Equations in H_∞ Control Problems

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Abstract

Algebraic matrix Riccati equations of the form, $FP + PF^T - PRP + Q = 0$, are analyzed with reference to the stability of closed-loop system $F - PR$. Here F, R and Q are $n \times n$ real matrices with $R = R^T$ and $Q = Q^T \geq 0$ (nonnegative-definite). Such equations have been playing key roles in optimal control and filtering problems with $R \geq 0$, and also in the solutions of in H_∞ control problems with R taking the form $R = H_1^T H_1 - H_2^T H_2$. In both cases, an existence of stabilizing solution, i.e. the solution yielding asymptotically stable closed-loop system, is an important problem.

First, we briefly review the typical results when R is of definite form, namely either $R \geq 0$ as in LQG problems or $R \leq 0$. They constitute two extreme cases of Riccati equations arising in H_∞ control theory in the sense that they correspond respectively to the cases $H_2 = 0$ and $H_1 = 0$. Necessary and sufficient conditions are shown for the existence of nonnegative-definite or positive-definite stabilizing solution.

Secondly, we focus our attention on more general case where R is only assumed to be symmetric, which obviously includes the case for H_∞ control problems. Here, necessary conditions are established for the existence of nonnegative-definite or positive-definite stabilizing solutions. The results are established by employing consistently the so-called algebraic method based on an eigenvalue problem of a Hamiltonian matrix.

1. Introduction and Preliminaries

We consider algebraic matrix Riccati equations of the following form.

$$FP + PF^T - PRP + Q = 0 \tag{1}$$

where F, R and Q are $n \times n$ real matrices and it is assumed that $R = R^T$ and $Q = Q^T \geq 0$. The closed-loop system associated with eq.(1) is given as

$$F_c = F - PR \tag{2}$$

Notice that, in the typical H_∞-control problems, the matrix R takes the form of $R = H_1^T H_1 - H_2^T H_2$. Here we consider existence conditions of real nonnegative-definite or positive-definite stabilizing solutions of eq.(1), i.e., the real solution $P \geq 0$ or $P > 0$ such that F_c in eq.(2) is an asymptotically stable matrix.

As in the case of matrix Riccati equations for optimal regulator and Kalman filter, eq.(1) can be analyzed by the so-called algebraic method. Let A be the $2n \times 2n$ Hamiltonian matrix defined by

$$A = \begin{bmatrix} F & Q \\ R & -F^T \end{bmatrix}. \tag{3}$$

Solving the eigenvalue problem for A, arbitrary n eigenvalue eigenvector relation can be written as

$$A \begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} Y \\ X \end{bmatrix} \Lambda \tag{4}$$

where Y and X are $n \times n$ matrices consisting of upper and lower halves of eigenvectors, and $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$ is the matrix consisting of n eigenvalues of A. Then a solution of eq.(1) can be obtained as

$$P = YX^{-1} \quad \text{if} \quad |X| \neq 0 \tag{5}$$

In this case, the closed-loop system matrix in eq.(2) is expressed as

$$F_c = -X^{-T} \Lambda X^T. \tag{6}$$

It should be noted that the ordering of n eigenvalues and eigenvectors in eq.(4) is immaterial on computing the solution P by eq.(5). Also the following assertion holds in eq.(4):

$$\begin{aligned} X^*Y = Y^*X \quad &\text{if} \quad \lambda_i + \lambda_j^* \neq 0, \quad 1 \leq i,j \leq n \\ X^TY = Y^TX \quad &\text{if} \quad \lambda_i + \lambda_j \neq 0, \quad 1 \leq i,j \leq n \end{aligned} \tag{7}$$

For the development of theory, we introduce the similarity transformation for F and F^T,

$$FW = W\Sigma \tag{8}$$

$$F^TZ = Z\Sigma \tag{9}$$

Here W and Z can be chosen as

$$WZ^T = W^TZ = I \text{ } (n \times n \text{ identity matrix}) \tag{10}$$

The concepts of controllability, stabilizability, observability and detectability play important roles in characterizing solutions of matrix Riccati equations, and are presented below for convenience. Consider linear time-invariant control systems.

$$\begin{aligned} \dot{x}(t) &= Fx(t) + Gu(t), \ x(t_0) = x_0 \\ y(t) &= Hx(t) \end{aligned}$$

with x, u and y being system state, input and output vectors, respectively. Then.

Definition 1. *An eigenvalue λ of F is said to be (F,G)-uncontrollable ((H,F)-unobservable) if there exists an n-vector $w \neq 0$ such that $F^Tw = \lambda w$ and $G^Tw = 0$ ($Fw = \lambda w$ and $Hw = 0$). Otherwise λ is said to be (F,G)-controllable ((H,F)-observable.*

Definition 2. *A pair (F,G) is said to be controllable ((H,F) is said to be observable) if all the eigenvalues λ of F are (F,G)-controllable ((H,F)-observable).*

Definition 3. *A pair (F,G) is said to be stabilizable ((H,F) is said to be detectable) if all the eigenvalues λ of F with $\Re[\lambda] \geq 0$ are (F,G)-controllable ((H,F)-observable.*

Finally, since $Q \geq 0$, we set $Q = GG^T$ in the subsequent sections.

2. The Case $R \geq 0$ and $R \leq 0$

2.1. The Case $R \geq 0$

The case $R \geq 0$ corresponds to algebraic matrix Riccati equations in Kalman filters or optimal regulators, and has been studied in detail. Clearly, from eqs.(4)-(6), the stabilizing solution yielding asymptotically stable closed-loop system in eq.(2) is obtained as $P = YX^{-1}$ with $\Re[\lambda_i(\Lambda)] > 0$, $i = 1, 2, \cdots, n$. Letting $R = H^TH$, its existence is established (Kucera 1972) as

Lemma 1. *There exists a stabilizing solution $P \geq 0$ if and only if A has no eigenvalue on the imaginary axis and the pair (H,F) is detectable.*

For positive-definite stabilizing solution, we have (Kano 1987)

Lemma 2. *There exists a stabilizing solution $P > 0$ if and only if the pair $(-F,G)$ is stabilizable and (H,F) is detectable.*

It is known that the following conditions (C1) and (C2) are equivalent.

(C1) A has no eigenvalue on the imaginary axis.

(C2) All the eigenvalues of F on the imaginary axis, if they exist, are (F,G)-controllable and (H,F)-observable.

Thus, the condition (C1) appearing in these lemmas can be replaced by (C2), which in view of (H,F)-detectability further replaced by

(C3) All the eigenvalues of F on the imaginary axis, if they exist, are (F,G)-controllable.

Therfore Lemma 1 can be rewritten as

Corollary 1. *There exists a stabilizing solution $P \geq 0$ if and only if the pair (H,F) is detectable and all the eigenvalues of F on the imaginary axis are (F,G)-controllable.*

Several remarks are in order. In contrast to the stabilizing solution, we may consisder the so-called anti-stabilizing solution for which all the eigenvalues of the closed-loop system have positive real parts. An existence condition of such a solution is known. Moreover, if stabilizing and anti-stabilizing solutions, denoted respectively as P_s and P_a, exist, then it holds that $P_a \leq P \leq P_s$ for any real symmetric solution P. These solutions constitute lattice with P_s and P_a the maximum and minimum elements respectively.

2.2. The Case $R \leq 0$

Similarly as in Sec.2.1, we have the following existence condition for stabilizing solution (Kano and Nishimura 1993) by letting $R = -H^TH$.

Lemma 3. *There exists a stabilizing solution P if and only if A has no eigenvalue on the imaginary axis and the pair (H,F) is detectable.*

It is noted that the above condition is the same as in Lemma 1 except that the stabilizing solution is not necessarily nonnegative-definite in this case. An obvious example is a scalar equation $p^2 + 3p + 2 = 0$.

Existence conditions for nonnegative-definite and positive-definite P can then be derived as (Nishimura 1990)

Lemma 4. *There exists a stabilizing solution $P \geq 0$ if and only if A has no eigenvalue on the imaginary axis and F is asymptotically stable.*

Lemma 5. *There exists a stabilizing solution $P > 0$ if and only if A has no eigenvalue on the imaginary axis, F is asymptotically stable, and (F, G) is controllable.*

If F is asymptotically stable, then the condition (C1) in Sec.2.1 is equivalent to

(C4) $\|H(sI - F)^{-1}G\|_\infty < 1$.

Here $\|\cdot\|_\infty$ denotes the H_∞ norm. Then the corresponding statements in Lemmas 4 and 5 can be equivalently replaced by (C4) (but not in Lemma 3), and hence

Corollary 2. *There exists a stabilizing solution $P \geq 0$ if and only if F is asymptotically stable and $\|H(sI - F)^{-1}G\|_\infty < 1$.*

Corollary 3. *There exists a stabilizing solution $P > 0$ if and only if F is asymptotically stable, $\|H(sI - F)^{-1}G\|_\infty < 1$, and (F, G) is controllable.*

It is further noted that, unlike the case in Sec.2.1, the equivalence of (C1) and (C2) no longer holds when $R \leq 0$. An obvious example is a scalar case with $F = 0, H = 1$ and $G = 1$.

Existence conditions for anti-stabilizing solution are also known, and in contrast to the case in Sec.2.1, it holds that $P_s \leq P \leq P_a$ for any real, symmetric solution P.

3. Main Results

For the case where R is not necessarily of definite matrix, the main result of this paper is stated as follows.

Theorem 1. *A necessary condition for eq.(1) to possess the stabilizing solution $P \geq 0$ is that $\Re[\lambda_i(A)] \neq 0$, $i = 1, 2, \cdots, 2n$, and $w^* R w > 0$ for all $w \neq 0$ such that $Fw = \lambda w$ with $\Re[\lambda] \geq 0$.*

(Proof) We assume that there exists a stabilizing solution $P \geq 0$. Such a solution is then obtained as $P = YX^{-1}$ where Y and X in eq.(4) are associated with Λ such that $\Re[\lambda_i(\Lambda)] > 0$. Then we have

$$FY + GG^T X = Y\Lambda \tag{11}$$

$$HY - F^T X = X\Lambda \tag{12}$$

where $\Re[\lambda_i(\Lambda)] > 0$.

Thus A must have n eigenvalues with positive real parts, and consequently the other n eigenvalues have negative real parts. Hence $\Re[\lambda_i(A)] \neq 0$, $i = 1, 2, \cdots, 2n$ is necessary. In order to derive the remaining condition, we assume that there exists $w \neq 0$ such that

$$Fw = \lambda w, \quad \Re[\lambda] \geq 0. \tag{13}$$

In eqs.(11) and (12), if Y is singular, Y can be set without loss of generality as $Y = [Y_1,\ 0]$ with $n \times m$ $(1 \leq m < n)$ matrix Y_1 being full rank, i.e. $\text{rank}[Y_1] = m$ (see Appendix A). Here notice that the case $Y = 0$ is excluded since otherwise F has only (uncontrollable) eigenvalues with negative real parts, contradicting the present problem setting. Let the nonsingular matrix X be partitioned accordingly as $X = [X_1,\ X_2]$. Moreover for the sake of convenience, we include in the above the case where Y is nonsingular with the understanding that $m = n$ and hence $1 \leq m \leq n$.

Now it can be shown (see Appendix B for the proof) that there exists an n-vector $d \neq 0$ such that

$$Pd = w. \tag{14}$$

Using eqs.(1) with $Q = GG^T$, (13) and (14), we then get

$$\begin{aligned} 0 &= d^*(FP + PF^T - PRP + GG^T)d \\ &= d^*Fw + w^*F^Td - w^*Rw + d^*GG^Td \\ &= (\lambda + \lambda^*)d^*Pd - w^*Rw + d^*GG^Td \end{aligned} \tag{15}$$

Since $\lambda + \lambda^* \geq 0$ and $d^*Pd \geq 0$, we obtain

$$w^*Rw = (\lambda + \lambda^*)d^*Pd + d^*GG^Td \geq 0 \tag{16}$$

Moreover $d^*Pd > 0$, since otherwise $d^*Pd = 0$ implies $Pd = 0$ and hence $w = 0$ by eq.(14).

Now assuming $w^*Rw = 0$, eq.(16) yields $\Re[\lambda] = 0$ and $G^Td = 0$. From eq.(11), we then get

$$FP + GG^T = Y\Lambda X^{-1} \tag{17}$$

leading to

$$Y\Lambda X^{-1}d = FPd = Fw = \lambda w = \lambda Pd = \lambda YX^{-1}d. \tag{18}$$

Hence we get

$$Y(\Lambda - \lambda I)X^{-1}d = 0 \tag{19}$$

Since $Y = [Y_1,\ 0]$, partitioning Λ and $\hat{d} = X^{-1}d$ accordingly results in

$$[Y_1 \quad 0]\begin{bmatrix} \Lambda_1 - \lambda I & 0 \\ 0 & \Lambda_2 - \lambda I \end{bmatrix}\begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix} = 0 \tag{20}$$

or

$$Y_1(\Lambda_1 - \lambda I)\hat{d}_1 = 0 \qquad (21)$$

Here noting that $\Re[\lambda(\Lambda_1)] > 0$, $\Re[\lambda] = 0$ and $\hat{d}_1 \neq 0$ (see Appendix B), we see that Y_1 is not of full rank, a contradiction. Thus we get $w^* R w > 0$. (Q.E.D.)

Corollary 3. *A necessary condition for eq.(1) to possess the stabilizing solution $P > 0$ is that $\Re[\lambda_i(A)] \neq 0$, $i = 1, 2, \cdots, 2n$, $w^* R w > 0$ for all $w \neq 0$ such that $Fw = \lambda w$ with $\Re[\lambda] \geq 0$, and all the eigenvalues λ of F with $\Re[\lambda] < 0$ are (F, G)-controllable.*

(Proof) From Theorem 1, we only need to prove that all the eigenvalues λ of F with $\Re[\lambda] < 0$ must be (F, G)-controllable.

On the contrary, let λ ($\Re[\lambda] < 0$) be an uncontrollable eigenvalue of F. Then, $F^T x = \lambda x$ and $G^T x = 0$ for $x \neq 0$, and it holds that $A \begin{bmatrix} 0 \\ x \end{bmatrix} = (-\lambda) \begin{bmatrix} 0 \\ x \end{bmatrix}$ for $\Re[-\lambda] > 0$. Hence Y is of the form of $Y = [0, \ Y_1]$ and $P = YX^{-1}$ cannot be positive-definite. (Q.E.D.)

If $R = H^T H \geq 0$ in Theorem 1, the condition $w^* R w > 0$ reduces to $Hw \neq 0$ for all $w \neq 0$ such that $Fw = \lambda w$ with $\Re[\lambda] \geq 0$. This is equivalent to that the pair (H, F) is detectable, and coincides with the necessary and sufficient condition in Lemma 1. On the other hand, if $R = -H^T H \leq 0$, then $w^* R w > 0$ can never be satisfied, implying that all the eigenvalues of F have negative real parts, or F is asymptotically stable. This also coincides with the condition in Lemma 4. The similar arguments hold also for positive-definite soutions.

4. Conclusions

We studied stabilizing solutions of algebraic matrix Riccati equations of the form, $FP + PF^T - PRP + GG^T = 0$.

First, several typical results were reviewed for the cases where the matrix R takes definite forms, namely either $R = H^T H \geq 0$ or $R = -H^T H \leq 0$. In either case, existence conditions of nonnegative-definite or positive-definite stabilizing solution are obtained as necessary and sufficient conditions. Also similarities and differences among these two cases were discussed.

Then we considered more general cases where R is only assumed to be symmetric. Obviously this includes the Riccati equations arising in H_∞ control theories, namely the case with $R = H_1^T H_1 - H_2^T H_2$. Specifically, we established, in Theorem 1 and Corollary 3, necessary conditions for the existence of nonnegative-definite or positive-definite stabilizing solutions. These conditions coinside with the necessary and sufficient conditions in Sec. 2 when R is of definite form.

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Appendix A

Here we examine a structure of matrix Y in eqs.(11) and (12) with $\Re[\lambda_i(\Lambda)] > 0$ for all $i = 1, 2, \cdots, n$. Specifically we show that

(i) If all the eigenvalues λ of F with $\Re[\lambda] < 0$, if it exists, are (F, G)-controllable, then $|Y| \neq 0$.

(ii) If all the eigenvalues λ of F are such that $\Re[\lambda] < 0$ and they are (F, G)-uncontrollable, then $Y = 0$.

(iii) In other cases, Y can be assumed to be of the form $Y = [Y_1, \ 0]$ where $n \times m$ matrix $(1 \leq m < n)$ Y_1 is of full rank, i.e. $\mathrm{rank}[Y_1] = m$.

We only prove the case (iii), since (i) and (ii) can then be deduced readily. This is the case where there exists at least one, but not all, eigenvalue λ of F such that $\Re[\lambda] < 0$ and λ is (F, G)-controllable. Then, such an eigenvalue yields the eigenvalue $-\lambda$ (hence $\Re[-\lambda] > 0$) of A with the corresponding eigenvector of the form $\begin{bmatrix} 0 \\ x \end{bmatrix}$. Now assume that all such eigenvectors are arranged in $\begin{bmatrix} Y \\ X \end{bmatrix}$ as $\begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} Y_1 & 0 \\ X_1 & X_2 \end{bmatrix}$, where Y_1 and X_1 are $n \times m$ $(1 \leq m < n)$ matrices. We then show that $\mathrm{rank}[Y_1] = m$. Let the matrix Λ be partitioned in accordance with Y and X, namely $\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$.

Then from eqs.(11) and (12), we get

$$FY_1 + GG^T X_1 = Y_1 \Lambda_1 \qquad (A1)$$

$$RY_1 - F^T X_1 = X_1 \Lambda_1 \qquad (A2)$$

Assuming, on the contrary, that Y_1 is not of full rank, there exists an m-vector $b_1 \neq 0$ such that $Y_1 b_1 = 0$. Thus eqs.(A1) and (A2) yield

$$GG^T X_1 b_1 = Y_1 \Lambda_1 b_1 \qquad (A3)$$

$$-F^T X_1 b_1 = X_1 \Lambda_1 b_1 \qquad (A4)$$

We then obtain

$$b_1^* X_1^* GG^T X_1 b_1 = b_1^* X_1^* Y_1 \Lambda_1 b_1 = b_1^* Y_1^* X_1^* \Lambda_1 b_1 = 0. \qquad (A5)$$

yielding $G^T X_1 b_1 = 0$ and $Y_1 \Lambda_1 b_1 = 0$. Then since $(Y_1^* Y_1) b_1 = 0$ and $(Y_1^* Y_1) \Lambda_1 b_1 = 0$, there exists a vector $\hat{b}_1 \neq 0$ such that $(Y_1^* Y_1) \hat{b}_1 = 0$ or $Y_1 \hat{b}_1 = 0$ and $\Lambda_1 \hat{b}_1 = \mu \hat{b}_1$. This shows that μ is an eigenvalue of Λ_1

hence $\Re[\mu] > 0$. From eqs.(A3) and (A4) with b_1 replaced by $\hat{b}_1$, we get

$$\begin{aligned} G^T(X_1\hat{b}_1) &= 0 \\ F^T(X_1\hat{b}_1) &= -X_1\Lambda_1\hat{b}_1 = -\mu(X_1\hat{b}_1) \end{aligned} \qquad (A6)$$

Noting that $X_1\hat{b}_1 \neq 0$, eq.(A6) shows that F has an uncontrollable eigenvalue $-\mu$ ($\Re[-\mu] < 0$), and that the eigenvector of A associated with the eigenvalue μ takes the form $\begin{bmatrix} 0 \\ X_1\hat{b}_1 \end{bmatrix}$.

Thus, with the understanding that such an eigenvector is already included in the second portion of $\begin{bmatrix} Y \\ X \end{bmatrix}$, namely $\begin{bmatrix} 0 \\ X_2 \end{bmatrix}$, we can assume without loss of generality that Y_1 is of full rank. (Q.E.D.)

Appendix B

We prove the following: If the stabilizing solution P exists, then there exists a vector $d \neq 0$ satisfying

$$Pd = w \qquad (B1)$$

for any eigenvector w of F corresponding to an eigenvalue λ which is either $\Re[\lambda] \geq 0$ or else (F, G)-controllable.

First consider the case (iii) in Appendix A, namely $Y = [Y_1,\ 0]$ with $n \times m$ matrix Y_1 being full rank, and X is partitioned accordingly as $X = [X_1,\ X_2]$. Now, in view of Y, F has $n - m$ (F, G)-uncontrollable eigenvalues with negative real parts. Hence, W, Z and Σ in eqs.(8) and (9) can be partitioned as $W = [W_1,\ W_2]$, $Z = [Z_1,\ Z_2]$ and $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ so that Σ_2 contains these uncontrollable eigenvalues.

Then we see that w in eq.(B1) must be one of the columns in matrix W_1, say i-th column w_i ($1 \leq i \leq n-m$). Now, it suffices to prove the existence of a vector $\hat{d} = X^{-1}d \neq 0$ such that $Y\hat{d} = w$ since $P = YX^{-1}$, or equivalently an existence of m-vector $\hat{d}_1$ such that $Y_1\hat{d}_1 = w$, since

$$w = Y\hat{d} = [Y_1 \quad 0]\begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix} = Y_1\hat{d}_1 \qquad (B2)$$

Notice that $X_2 = Z_2$ and $X_2^T Y_1 = 0$ by $X^TY = Y^TX$, and hence $Z_2^T Y_1 = 0$ yielding

$$Y_1 = WZ^TY_1 = (W_1Z_1^T + W_2Z_2^T)Y_1 = W_1(Z_1^TY_1) \quad (B3)$$

Thus eq.(B2) can be rewritten as

$$w = Y_1\hat{d}_1 = W_1(Z_1^TY_1)\hat{d}_1 \qquad (B4)$$

On the other hand, we have

$$Z^TY_1 = \begin{bmatrix} Z_1^T \\ Z_2^T \end{bmatrix} Y_1 = \begin{bmatrix} Z_1^TY_1 \\ 0 \end{bmatrix} \qquad (B5)$$

and since $\text{rank}[Z^TY_1] = m$ we obtain $|Z_1^TY_1| \neq 0$. Therefore, in eq.(B4), we can set

$$\hat{d}_1 = (Z_1^TY_1)^{-1}e_i \ (\neq 0) \qquad (B6)$$

where e_i is the m-vector with only nonzero element one in its i-th position.

Notice that the vector d in eq.(B1) is then given as

$$d = X\hat{d} = [X_1 \quad X_2]\begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix} = X_1(Z_1^TY_1)^{-1}e_i + X_2\hat{d}_2 \qquad (B7)$$

with $\hat{d}_2$ being any $(n-m)$-vector.

In the case (i) in Appendix B, eq.(B7) obviously degenerates to

$$d = X(Z^TY)^{-1}e_i = XY^{-1}Z^{-T}e_i = P^{-1}We_i = P^{-1}w_i \qquad (B8)$$

which can be readily deduced from eq.(B1). Notice that the case (ii) does not occur under the present assumption that w is an eigenvector of F corresponding to an eigenvalue which is either $\Re[\lambda] \geq 0$ or (F, G)-controllable. (Q.E.D.)

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