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A Calculation Method of Root Loci Band and its Applications to Robust Control System Design

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Abstract This paper presents a method to calculate the characteristic root areas and loci band of control systems with uncertainties. First, equations of boundary curves of root areas in the case of additive and multiplicative perturbation are derived. Then, an algorithm for the calulation of the array of closed curves is presented. When the upper bound of the absolute values of frequency responses for the uncertain part is also frequency-dependent, the frequency-dependent terms are included in the characteristic equation of the nominal system. This lead to the boundary equations of the root areas for control systems with frequency-dependent uncertainty. Numerical examples of the control systems with multiplicative perturbations including frequency-dependent terms are presented to verify this calculation method. Finally, its applications to the design of robust control systems, e.g., passive adaptive control systems are also discussed.

1 Introduction

In this paper, we present a method to calculate the characteristic root area and its array, i.e., root loci band of control systems with plant uncertainties. First, the method of calculating the characteristic root areas for additive and multiplicative perturbations is discussed. If the upper bound of the absolute values of frequency responses for the uncertain part is given, the equation of the boundary curves can be derived, and an algorithm for the numerical calculation of the array of closed curves can be presented.

A little adjustment is performed between the nominal system and the uncertain part particularly in the case of multiplicative perturbation, so that the characteristic root area is not too large, or in other word, boundary curves are not too conservative. When the upper bound of the absolute values of frequency responses for the uncertain part is also frequency-dependent, the frequency-dependent terms are included in the characteristic equation of the nominal system.

A simple design example of robust control, i.e., passive adaptive control system by using model reference is presented as an application of the root loci band.

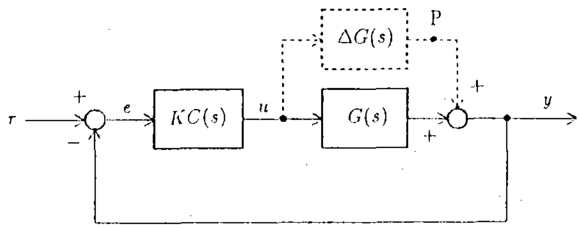

Fig.1 Feedback control system with uncertainties.

2 Root Loci Band for Control Systems with Uncertainties

2.1 Additive Perturbation

A control system with additive perturbation, e.g., interconnection to the other loops is considered as shown in Fig.1. The open loop transfer function from the viewpoint of P is given by

$$L(s) = \frac{\Delta G(s) KC(s)}{1 + KC(s)G(s)} , \tag{1}$$

where G, C, and K is plant, compensator and gain parameter respectively. Obviously, the robust stability condition can be written as

$$\|L(s)\|_\infty < 1 . \tag{2}$$

If the upper bound of the absolute values of frequency responses for the uncertain part is given by a function of real frequency ω as

$$|\Delta G(j\omega)| \le |\rho_a(j\omega)| , \tag{3}$$

then Eq.(2) is arranged by

$$|F_a(j\omega)| > |\rho_a(j\omega)| . \tag{4}$$

Where

$$F_a(s) = \frac{1 + KC(s)G(s)}{KC(s)} = \frac{N_a(s)}{D_a(s)} \qquad (5)$$

and any zero of Eq.(5) lies in the left-half s plane.

The robust stability condition Eq.(4) contains all properties of such control systems for complex frequencies s. However, it is more convenient to see how the array of boundary curves[1], which is refered to as the *root loci band* in this paper, varies with increasing gain K. So a method to calculate the array of the curves is mentioned below.

The locus of a zero s_i $(i = 1, 2, \cdots, n)$ of the following equation,

$$|F_a(s, K)| = 0 \qquad \text{and} \qquad |N_a(s, K)| = 0 \qquad (6)$$

for $K : 0 \to \infty$ is obviously a root locus, whereas for some positive constant $\bar{\rho}_a$ the array of closed curves satisfying

$$|F_a(s, K_l)| = \bar{\rho}_a > |\rho_a(s)|, \quad l = 1, 2, \cdots, \infty \qquad (7)$$

shows the boundary curves of the area containing characteristic roots of control systems. This inequality Eq.(7) corresponds to Rouché's theorem[2]. Though the right-side inequality of Eq.(7) should be also satisfied, the consideration of the inequality will be discussed later.

2.2 Multiplicative Perturbation

When the uncertain part in Fig.1 is represented by $\Delta G(s) = \Delta(s)G(s)$, it is possible to discuss control systems with multiplicative perturbation in the same way as those with additve perturbation. The open loop transfer function from the view-point of P is given by

$$L(s) = \frac{\Delta(s)KG(s)C(s)}{1 + KC(s)G(s)}. \qquad (8)$$

If the upper bound of the absolute values of frequency responses for the uncertain part is given by a function of real frequency ω as

$$|\Delta(j\omega)| \le |\rho_m(j\omega)| \,. \qquad (9)$$

then Eq.(2) is arranged by

$$|F_m(j\omega)| > |\rho_m(j\omega)| \,. \qquad (10)$$

Where

$$F_m(s) = \frac{1 + KC(s)G(s)}{KC(s)} = \frac{N_m(s)}{D_m(s)} \qquad (11)$$

and any zero of Eq.(11) lies in the left-half s plane.

In order to avoid conservativeness, a little adjustment is performed between the nominal system and the uncertain part particularly when dealing with multiplicative perturbation. If the parameter τ of uncertainty varies from 0 to $\tau_{\max}$, the modifying term by which to multiply the nominal system is as follows[2]

$$D_n(s, \tau_{\max}) = 1 + \frac{\Delta(s, 0) + \Delta(s, \tau_{\max})}{2} \,. \qquad (12)$$

Therefore, the modified term $\Delta(s)$ of uncertainty can be written as

$$\Delta^*(s, \tau_{\max}) = \frac{\Delta(s, 0) - \Delta(s, \tau_{\max})}{2 + \Delta(s, 0) + \Delta(s, \tau_{\max})} \,. \qquad (13)$$

3 Algorithm Obtaining Boundaries of Root Areas

3.1 Boundary Curves

The equation of the boundary curves corresponding to Eq.(7), in other words, *root contours* $\Gamma_{i,l}$ is generally represented by

$$f(\sigma, \omega, K_l) = |F(s, K_l)| - \bar{\rho}, \quad l = 1, 2, \cdots, \infty. \qquad (14)$$

Roots $s = (\sigma, \omega)$ of this equation are obtained sequentially by using Newton's algorithm together with the gradient method in the complex plane. Note that any boundary curve of Eq.(14) is a simple closed curve, i.e., a Jordan curve.

If it is necessary to confirm the validity of the inequality on the right side of Eq.(13), the equation of the boundary curve

$$g(\sigma, \omega, K_l) = |\rho(s)| - \bar{\rho}, \quad l = 1, 2, \cdots, \infty \qquad (15)$$

can be calculated for the confirmation.

The algorithm obtaining the closed boundary curves represented by Eq.(14) will be shown in several calculation steps. A course of computation is illustrated in Fig.2.

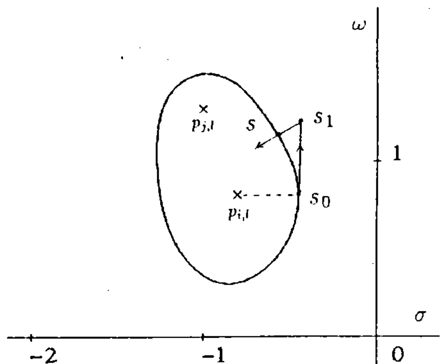

Fig.2 A root contour and its calculation.

3.2 Frequency-dependent Uncertainty

When the upper bound of the absolute values of frequency responses for the uncertain part is frequency-dependent, the frequency-dependent terms such as the right side of Eq.(10) can be included on the left side of the inequality. If the frequency-dependent 'radius' is given by

$$\rho(s) = \epsilon \cdot \frac{N_\rho(s)}{D_\rho(s)}, \qquad \epsilon \ : \ \text{const.}, \tag{16}$$

then the characteristic functions Eqs.(5) and (11) are as follows:

$$F^*(s) = \frac{N(s)}{D(s)} \cdot \frac{D_\rho(s)}{N_\rho(s)} \,. \tag{17}$$

Thus, the inequality corresponding to Eq.(7) is

$$|F^*(s, K_l)| = \bar{\rho}^* > |\epsilon| \,, \quad l = 1, 2, \cdots, \infty \,, \tag{18}$$

and Eq.(14) is modified as the following equation:

$$f(\sigma, \omega, K_l) = |F^*(s, K_l)| - \bar{\rho}^* = 0, \ l = 1, 2, \cdots, \infty. \tag{19}$$

In later examples, we will use Eq.(19) as the equation of the root contours.

3.3 Algorithm

The algorthm for the calculation of Eqs.(14) and (15) is as follows[2] :

(1) Let $i := 1$, $l := 1$ be the initial setting.

(2) Calculate root locus according to Eq.(6) from $K = K_{l-1}$ ($l = 1, 2, \cdots$) to the appropriate gain $K = K_l$, by using the two-dimensional Newton method. (If Eq.(14) is used, let $\bar{\rho} = 0$). When calculating the root locus, it is necessary to pay close attention to the vicinity of the points where roots are breakaway from real quantities to complex ones, or break-in from complex quantities to real ones[3], i.e., multiple roots, a node or a saddle point in Eq.(14). However, the method to calculate such points is not discussed here in detail.

(3) Proceed from the root $p_{i,l}$ of the nominal system for $K = K_l$ to a direction of zero argument and then calculate root $s_0 = (\sigma_0, \omega_0)$ of Eq.(14) by using the Newton method.

(4) Proceed from the point s_0 to the tangential direction (the orthogonal direction to ∇F) and at this time proceed from the point s_1 to the direction of $-\nabla F$. And then calculate root $s = (\sigma, \omega)$ of Eq.(14) by using the Newton method.

(5) Repeat step(3) until returning the vicinity of point s_0.

(6) Let $l := l + 1$ and repeat from step(1) until $K = K_{max}$ is satisfied.

(7) Let $l := l + 1$ and repeat from step(1) until $i = n$ is satisfied.

4 Numerical Examples

4.1 First-order Lag Uncertainty

Example 1 As an examle, consider a control system as shown in Fig.1 where plant and compensator are given by

$$G(s) = \frac{1}{s(1+s)}, \quad C(s) = \frac{1 + 0.5s}{1 + 0.2s} \,. \tag{20}$$

Suppose the plant uncertainties for multiplicative perturbation are first-order lag and defined as

$$D(s, \tau) = 1 + \Delta = \left\{ \frac{1}{1 + \tau s} \,\middle|\, 0 \le \tau \le \tau_{\max} \right\} \,. \tag{21}$$

The modifying term Eq.(12) is

$$D_n(s, \tau_{\max}) = \frac{1 + (\tau_{\max}/2)}{1 + \tau_{\max} s} \,, \tag{22}$$

and the modified term of uncertainty Eq.(13) is

$$\Delta^*(s, \tau_{\max}) = \frac{(\tau_{\max}/2)s}{1 + (\tau_{\max}/2)s}. \tag{23}$$

In case of $\tau_{\max} = 0.4$, the array of the boundary curves, i.e., root loci band is shown in Fig.3.

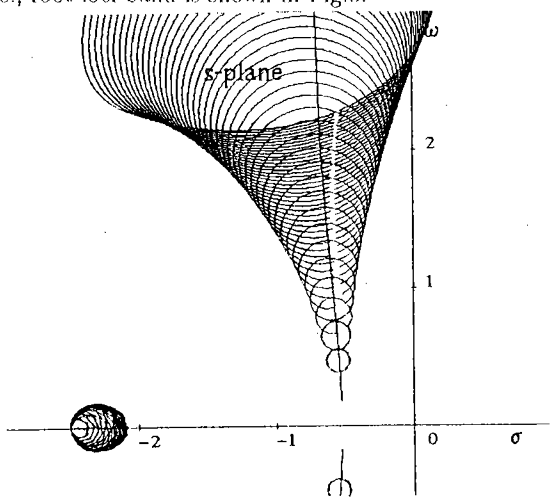

Fig.3 Root loci band of a control system with first-order lag uncertainty.

4.2 Transport Lag Uncertainty

Example 2 Consider transport lag uncertainty for multiplicative perturbation, that is,

$$D(s, \tau) = 1 + \Delta = \left\{ e^{-\tau s} \mid 0 \le \tau \le \tau_{\max} \right\} \,. \tag{24}$$

Suppose plant and compensator are the same as those in Example 1. For $\tau_{\max} = 0.4$, the array of the boundary curves, i.e., root loci band of dominant poles is shown in Fig.4.

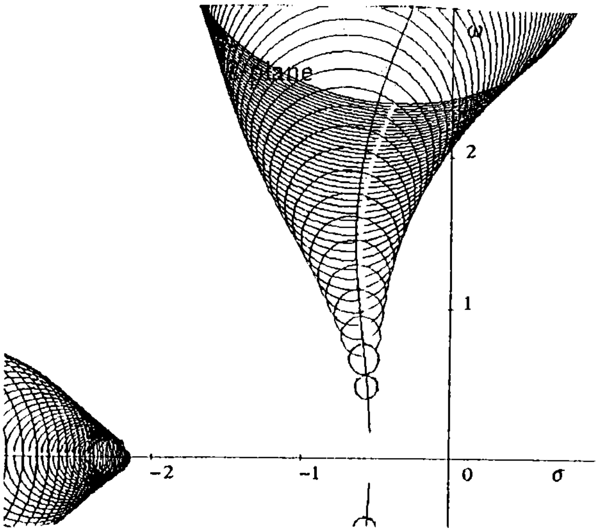

Fig.4 Root loci band of a control system with transport lag uncertainty.

5 Applications to Robust Control System Design

The block diagram of model-referenced passive adaptive control system is as shown in Fig.5. Where $G(s)$, $G_m(s)$, $C(s)$ and K is plant, plant model, compensator and gain parameter respectively. And $H(s)$ is some class of feedback compensators determined by the inverse function of $G_m(s)$. Note that $G_m(s)$ should be selected for the nominal plant $G(s)$.

It is well known that this type of control system is low-sensitive and robust to disturbances and/or plant uncertainties[4].

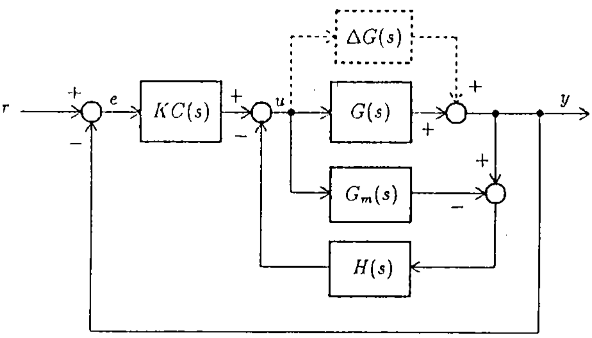

Fig.5 Model-referenced passive adaptive control system.

Example 3 When the uncertainties for multiplicative perturbation are first-order lag and defined as Eq.(21), $D(s,\tau)$ is modified as follows:

$$D(s,\tau)=\left\{\frac{(2+\tau s)(1+\tau_c s)}{(1+\tau s)(1+\tau_c s)+1}\ \middle|\ 0\le\tau\le\tau_{\max}\right\}, \tag{25}$$

where τ is uncertain or varying parameter and τ_c is the time constant of lag factor to realize the inverse function of $G_m(s)$. As assumed in Example 1, plant is given by $G(s)=\dfrac{1}{s(1+s)}$. Now compensator is not used, that is, $C(s)$=1. For $\tau_{\max}=0.4$, the array of the boundary curves, i.e., root loci band of dominant poles is shown in Fig.6.

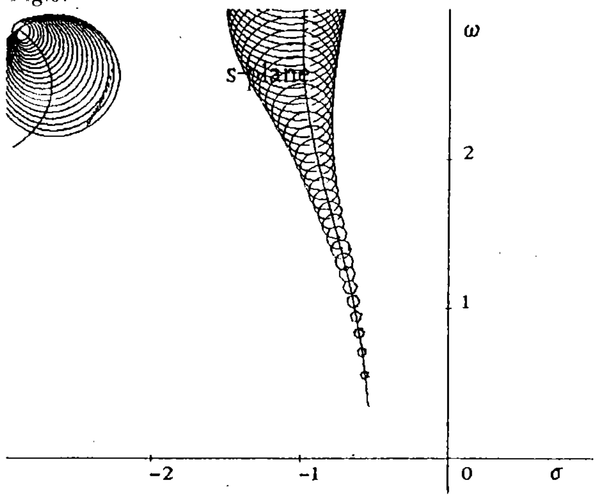

Fig.6 Root loci band of a passive adaptive control system with first-order lag uncertainty.

These numerical examples show the applicability of this method to the robust control systems design.

6 Conclusion

In this paper, we present a method to calculate the characteristic root areas and its algorithm. The array of the boundary curves is calculated easily on a workstation and gives an aid to control system designers. Therefore, the method proposed in this paper will be useful as the visualized method when designing robust control systems.

References

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