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On a sensitivity of optimal solutions in fuzzy mathematical linear programming problem

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Abstract:

The authors have been devoted to researches on fuzzy theories and their applications, especially control theory and application problems, for recent years.

In this paper, the authors present results on a comparison of optimal solutions between ones of an ordinary-typed mathematical linear programming problem(O.M.L.P. problem) and ones of a Zimmerman-typed fuzzy mathematical linear programming problem (F.M.L.P. problem), and comment about the sensitivity (differences and fuzziness on between O.M.L.P. problem and F.M.L.P. problem) on optimal solutions of these mathematical linear programming problems.

1. Introduction

A thesis on "Fuzzy set" presented by L.A.Zadeh made presentation first in 1965, and it was seemed that fuzzy theories constructed mainly by fuzzy sets, fuzzy logic(reasoning) and fuzzy measurements and integral etc. were systematized in the 1970's.[1] But, the affairs on fuzzy theoretical applications did not develop parallel with their theories. Now, as it was called 1985 "Fuzziness applications' new year", a boom on fuzziness applications came out in the middle years of 1980's and after, and in the 1990's, and especially in Asia areas.

In this paper, based on our reseaches on fuzzy theories and their applications[2)-5)] for recent several years, we will think an optimal decision -making problem under the fuzzy conditions : the fuzzy mathematical programming problem, which is made a linear-typed objective function optimized (maximized or minimized) under given, some linear-typed constraints. And it will be reported that judging from our theoretical formulations and our numerical examples of this problem, an existence and a sensitivity on optimal solutions of both an ordinary linear programming problem(O.M.L.P. problem) and a fuzzy mathematical linear programming problem(F.M.L.P. problem) are relative but different interestedly, that is, in the former problem, there exist optimal solutions within the feasible region of relaxed-constraits strictly, if the restrictions are relaxed; on the other hand, in the latter problem, there may exist optimal solutions without the feasible region of restrictions, and within the region added the shere which the degree of membership function of restrictions is between 1 and 0, and their locations may be fuzzy and the value of an objective function may increase fuzzily, not more than that in the former problem.

2. A sensitivity on optimal solutions in F.M.L.P. problem

2.1 A sensitivity in O.M.L.P. problem

2.1.1 Sensitivity analysis [6]

We will think the next linear-typed equations:

$$Ax=b,\quad A=[a_{ij}]\in R^{n\times n},\quad b=[b_i],\quad x=[x_i]\in R^n \qquad (1)$$

The values of components on A, b in (1) may be uncertain. For example, if (1) is represented the relations of an electri circuit model, we have to think that the values on A, b will be having a variance as the values on elements costructed by electrical circuit. Therefore, it will be needed to clear that how uncertainty of the values on A, b is related to uncertainty of solutions x . So, 'Analysis of Sensitivity' will be needed.

We will assume that in (1), if A, b are changed $A\to A+\Delta A,\ b\to b+\Delta b$, the changes of solutions in (1) are $x\to x+\Delta x$ (x is true solutions in (1)). And we will try to estimate the Δx . So,

$$(A+\Delta A)(x+\Delta x)=b+\Delta b \qquad (2)$$

<u>Theorem</u>

Let $\|\cdot\|$ be arbitary norm defined on R^n . And we will be given operational norm on $R^{n\times n}$ with the same way. If $\|A^{-1}\|\|\Delta A\|<1$,

$$\frac{\|\Delta x\|}{\|x\|}\le\frac{\mathrm{cond}(A)}{1-\mathrm{cond}(A)(\|\Delta A\|/\|A\|)}\left\{\frac{\|\Delta A\|}{\|A\|}+\frac{\|\Delta b\|}{\|b\|}\right\} \qquad (3)$$

Here, $\mathrm{cond}(A)\equiv\|A^{-1}\|\|A\|=\mathrm{cond}(A^{-1})\ge1$

is called condition number of A.

Proof)

We will show the existence of $(A+\Delta A)^{-1}$, first thing, and will solve Δx in (1) and (2), and will estimate $\|\Delta x\|$.

$$A+\Delta A=A(I+A^{-1}\Delta A)$$

(Existence of A^{-1} is assumed)

So, to show the existence of $(A+\Delta A^{-1})$, it is sufficient that we will be able to show the existence of $(I+A^{-1}\Delta A)^{-1}_{.}$ That is, we will have to show $\|A^{-1}\Delta A\|<1$.For this, will be verfied by a given condition: $\|A^{-1}\|\|A\|<1$.

Now, we will solve Δx in (1) and (2),

$$\Delta x=(A+\Delta x)^{-1}(-\Delta Ax+\Delta b)$$

$$=\{A(I+A^{-1}\Delta A)\}^{-1}(-\Delta Ax+\Delta b)$$

$$=(I+A^{-1}\Delta A)^{-1}A^{-1}(-\Delta Ax+\Delta b)$$

If we take a norm in both hands,

$$\|\Delta x\|\le\|(I+A^{-1}\Delta A)^{-1}\|\|A^{-1}\|(\|\Delta A\|\|x\|+\|\Delta b\|)$$

$$\le\frac{\|A^{-1}\|}{1-\|A^{-1}\|\|\Delta A\|}(\|\Delta A\|\|x\|+\|\Delta b\|)$$

And devided by $\|x\|$,

$$\frac{\|\Delta x\|}{\|x\|}\le\frac{\|A^{-1}\|\|A\|}{1-\|A^{-1}\|\|\Delta A\|}\left\{\frac{\|\Delta A\|}{\|A\|}+\frac{\|\Delta b\|}{\|A\|\|x\|}\right\}$$

Here, if we use the relation $\|A\|\|x\|\ge\|b\|$ derived by taking a norm in $Ax=b$, we will obtain the formulation in this theorem.

And, if we take a norm in $AA^{-1}=I$, $\|A\|\|A^{-1}\|\ge\|I\|=1$. Here, $\|I\|=1$ is given by the definition of operational norm. Therefore, $\mathrm{cond}(A)\ge1$. Q.E.D.

We will point out the next two facts:

1) it is possible that an inequality in (3) becomes an equality.
2) it is possible that when a condition number is large, some components in x receive large relational changes, even if components of are changed slightly.

2.1.2 O.M.L.P. Problem

In this paper, we will give the next distribution problem.

<u>Problem</u>

We will think that by using two machines M_1, M_2, two-kinds of products A, B are producted.

Hours needed to make a piece one of products by each machine, profits on a piece one of products, and a limitation of hours on each machine, are given in <u>Table</u>.

Under these conditions, obtain optimal volumes of products such that total profits may maximized.

Let x_1, x_2 be productive volumes of products A, B. We will obtain the next formulations:

Constraits:

$$x_1 + 4x_2 \leqq 8$$
$$3x_1 + 4x_2 \leqq 12$$
$$x_1,\ x_2 \geqq 0$$

Objective function(total profits);

$$z = -x_1 - 2x_1 \quad \text{, minimized.}$$

$$\hat{x}_1 = 2,\ \hat{x}_2 = 1.5,\ \hat{z} = -5$$

And we will obtain optimal solutions by using Program of Prof. Tone[7]. (See Fig.1)

Now, we will call this problem and solutions original ones.

2.1.3 Sensitivity on upper-limited values of constraints in an O.M.L.P. problem

We will think a sensitivity problem typed by the relaxed of upper-limited values of constraints in an O.M.L.P. problem.

In a F.M.L.P. problem, a relaxing of conditios on both constraints and a objective function, but in this section, a relaxing of upper-limited values of constraints are thought, and their widths are a half of it on a relaxing of a F.M.L.P. problem. (See Fig.2)

2.1.4 Numerical example

Upper-limited values of constraints are relaxed a half widths of a fuzzy cases, so the next sensitivity problem are obtained:

$$\begin{aligned} &\min \quad -(x_1 + 2x_2) \\ &\text{s.t.} \begin{cases} x_1 + 4x_2 \leqq 9 \\ 3x_1 + 4x_2 \leqq 13.5 \end{cases} \\ &\qquad x_1 \geqq 0,\ x_2 \geqq 0\ . \end{aligned}$$

We will obtain optimal solutions;

$$\hat{x}_1 = 2.25,\ \hat{x}_2 = 1.6875,\ \hat{z} = -5.625$$

and a profit increases 12.5 %, compared with the original ones. In this problem, a location of optimal solutions $\hat{x}_1, \hat{x}_2$ is the neighbourhood(near) in the original ones and within the relaxed-sphere of constraints. The value of objective function z is not so small.

2.2 A sensitivity in F.M.L.P. problem

2.2.1 Optimal decision-making under a fuzzy environment

For the decision-making that a fuzzy objective function and fuzzy constraints are given on the set X which is costructed by alternative plans, Bellman & Zadeh[8] proposed and defined the next:

a fuzzy objective function; O and fuzzy constraints; C are fuzzy sets on the set X which is constructed by alternative plans that is characterized by their membership function:

$$\mu_O : X \rightarrow [0,\ 1] \qquad (4)$$
$$\mu_C : X \rightarrow [0,\ 1] \qquad (5)$$

and, the unification of between a fuzzy objective function; O and fuzzy constraints; C, that is, a fuzzy making-decision; D made by thinking that D is satisfied both O and C at the same time, is defined by the intersection set of O and C,

$$D = O \cap C \qquad (6)$$
$$\mu_D(x) = \min(\mu_O(x),\ \mu_C(x)) \qquad (7)$$

Here, μ_D is denoted membership function of D.

To obtain an optimal fuzzy making-decision D*, maximized decision-making method(to select x which is maximized the values of a fuzzy making-decision D's membership function $\mu_D(x)$) was proposed: that is,select x* such that

$$\begin{aligned} \mu_D(x^*) &= \max_{x \in X} \mu_D(x) \\ &= \max_{x \in X}(\min(\mu_O(x),\ \mu_C(x)) \end{aligned} \qquad (8)$$

Now, these x* may not exist, and may exist infinitely.

2.2.2 F.M.L.P. problem with a fuzzy function and fuzzy constraints

Let $c = (c_1, c_2, \cdots, c_n)$, $x = (x_1, x_2, \cdots, x_n)^T$ be n-dimensional vectors, $b = (b_1, b_2, \cdots, b_m)^T$ be an m-dimensional vector, and $A = (a_{ij})$ be an m×n matrix. Here, T is denoted transpose.

For O.M.L.P. problem (9):

$$\begin{aligned} &\min \quad z = c \cdot x \\ &\text{s.t.} \quad A \cdot x \leqq b \\ &\qquad\quad x \geqq 0 \end{aligned} \qquad (9)$$

Zimmermann[9] proposed, based on M.L.P. problem,

$$\begin{aligned} c\cdot x &\lesssim z_0 \\ A\cdot x &\lesssim b \\ x &\geqq 0 \end{aligned} \tag{10}$$

and, thought of the F.M.L.P. problem that the value of objective function c·x is nearly smaller than z_0, and the values of constraints A·x are nearly smaller than b.

Therefore,

$$\begin{aligned} B\cdot x &\lesssim b' \\ x &\geqq 0 \end{aligned} \tag{11}$$

where,

$$B=\begin{bmatrix} c \\ A \end{bmatrix},\quad b'=\begin{bmatrix} z_0 \\ b \end{bmatrix}. \tag{12}$$

In fuzzy inequality $B\cdot x \lesssim b'$,especially i-th inequality:

$$(B\cdot x)_i \lesssim b'_i\ . \tag{13}$$

For $i=0,\ 1,\ \cdots,\ m$,

a linear-typed membership function: $\mu_i((B\cdot x)_i)$ is given the next;

$$\mu_i((B\cdot x)_i)=\begin{cases} 1 & ;\ ((B\cdot x)_i)\leqq b'_i \\ 1-((B\cdot x)_i)-b'_i)/d_i & ;\ b'_i\leqq((B\cdot x)_i)\leqq b'_i+d_i \\ 0 & ;\ ((B\cdot x)_i)\geqq b'_i+d_i \end{cases} \tag{14}$$

and represents "fuzziness" of a decision-maker. (See Fig.3)

Here, the value of d_i is set subjectively by a decision-maker.

To this problem (10), we will adopt Bellman & Zadeh maximized decision-making method. So, the problem (10) will convert into the problem which chooses x* such that

$$\mu_D(x^*)=\max_{x\geq 0}\ \min_{0\leq i\leq m}\{(B\cdot x)_i\} \tag{15}$$

that is, the problem which makes minimum membership function values maximized in x≧0 and chooses x*≧0.

Therefore, let $b''_i=b'_i/d_i,\ (B'\cdot x)_i=(B\cdot x)_i/d_i$, (16)

(15) will change the next:

$$\mu_D(x^*)=\max_{x\geq 0}\ \min_{0\leq i\leq m}\{1+b''_i-(B'\cdot x)_i\} \tag{17}$$

so, this problem will convert into the next mathematical programming problem:

$$\begin{aligned} &\max\ \lambda \\ &\text{s.t.}\ \ \lambda\leqq 1+b''_i-(B'\cdot x)_i \\ &\qquad x\geqq 0 \end{aligned} \tag{18}$$

2.2.3 A sensitivity on optimal solutions in F.M.L.P. problem

For this F.M.L.P. problem, we will give the next conditions.

That is, for an objective function:

we will give the width d_Z placed between -5 such that $\mu=0:-5+d_Z/2,\ \mu=1:-5-d_Z/2$,

for the constraints:

we will give the width d_{M1} such that

$\mu=0:8+d_{M_1},\ \mu=1:8$, with respect to M_1,

and the width d_{M2} such that

$\mu=0:12+d_{M_2},\ \mu=1:12$, with respect to M_2.

Therefore, this problem will change into the next: for given the widths d_Z, d_{M1} and d_{M2},

constraint with respect to M_1,

$$\lambda\leqq 1+8/d_{M_1}-(1/d_{M_1})x_1-(4/d_{M_1})x_2$$

so, $(1/d_{M_1})x_1+(4/d_{M_1})x_2+\lambda\leqq 1+8/d_{M_1}$.

constraint with respect to M_2,

$$\lambda\leqq 1+12/d_{M_2}-(3/d_{M_2})x_1-(4/d_{M_2})x_2$$

so, $(3/d_{M_2})x_1+(4/d_{M_2})x_2+\lambda\leqq 1+12/d_{M_2}$.

objective function,

$$\lambda\leqq 1+(-5-d_Z/2)-(-1/d_Z)x_1-(-2/d_Z)x_2$$

so, $(1/d_Z)x_1+(2/d_Z)x_2-\lambda\geqq 4+d_Z/2$.

That is,

$$\begin{aligned} &\max\ \lambda \\ &\text{s.t.}\ \begin{cases} (1/d_{M_1})x_1+(4/d_{M_1})x_2+\lambda\leqq 1+8/d_{M_1} \\ (3/d_{M_2})x_1+(4/d_{M_2})x_2+\lambda\leqq 1+12/d_{M_2} \\ (1/d_Z)x_1+(2/d_Z)x_2-\lambda\geqq 4+d_Z/2. \end{cases} \\ &\qquad x_1\geqq 0,\ x_2\geqq 0 \end{aligned}$$

Now, this problem can be solved by our modified program of Tone's one.

2.2.4 Numerical examples

The next three numerical examples and comments are given.

Example 1:

Let an objective function; $\mu=0$: -4.5, $\mu=1$: -5.5, $d_z=1$,

constraints;

M_1, $\mu=0$: 10, $\mu=1$: 8, $d_z=2$

M_2, $\mu=0$: 15, $\mu=1$: 12, $d_z=3$.

This example is the next:

$$\max \ \lambda$$
$$\text{s.t.} \begin{cases} (1/2)x_1+2x_2+\lambda \leqq 5 \\ x_1+(4/3)x_2+\lambda \leqq 5 \\ x_1+2x_2-\lambda \geqq 4.5 \end{cases}$$
$$x_1 \geqq 0,\ x_2 \geqq 0$$

We will obtain optimal solutions with our modified program of Tone.

$\hat{x}_1=2.11814$, $\hat{x}_2=1.5807$, $\hat{\lambda}=0.779534$, $\hat{z}=-5.27954$.

A profit increases 5.6 %, compared with the original ones. in this example, a location of optimal solutions x_1,x_2 is neighbourhood of the original ones, and without the shere of constraints in the original ones, but its location is within the additional-shere which the value of membership function is between 1 and 0, and is not on or out of the most outer boundary line (that is, the value of membership function is 0).

Example 2:

Let $d_z=1$, $d_{M_1}=2$, $d_{M_2}=2$.

This example is the next:

$$\max \ \lambda$$
$$\text{s.t.} \begin{cases} (1/2)x_1+2x_2+\lambda \leqq 5 \\ (3/2)x_1+2x_2+\lambda \leqq 7 \\ x_1+2x_2-\lambda \geqq 4.5 \end{cases}$$
$$x_1 \geqq 0,\ x_2 \geqq 0$$

We will obtain optimal solutions with our program.

$\hat{x}_1=2$, $\hat{x}_2=1.625$, $\hat{\lambda}=0.75$, $\hat{z}=-5.25$.

A profit increases 5% and x_1 is the same value.

Example 3:

Let $d_z=2$, $d_{M_1}=2$, $d_{M_2}=1$.

This example is the next:

$$\max \ \lambda$$
$$\text{s.t.} \begin{cases} (1/2)x_1+2x_2+\lambda \leqq 5 \\ 3x_1+4x_2+\lambda \leqq 13 \\ (1/2)x_1+x_2-\lambda \geqq 2 \end{cases}$$
$$x_1 \geqq 0,\ x_2 \geqq 0$$

We will obtain optimal solutions with our program.

$\hat{x}_1=1.81818$, $\hat{x}_2=1.72727$, $\hat{\lambda}=0.63634$, $\hat{z}=-5.27272$.

A profit increases 5.45 % and x_1 is smaller, x_2 is larger than those of the original ones.

The rate of increases in a profit in the case 2.2.4 is not always larger than it of the case of 2.1.4.

3. Conclusion

To the distribution problem which is formulated by a F.M.L.P. problem that is a fuzzy objective function and costraints, it is reported of the characteristics of optimal solutions and the value of objective function, and the sensitivity of 2.1.2↔2.1.4, 2.1.4↔2.2.4 and 2.1.4↔2.2.4 comparatively. It is easier to catch the result of sensitivity in 2.1.4, but 2.2.4 is not easier, fuzzy than linear-typed 2.1.4.[10]

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機械 \ 製品	A	B	制限 (時間)
M_1	1	4	8
M_2	3	4	12
利益(万円/個)	1	2	

Table Values on machines M_1, M_2; products A, B; limitations and profits per piece one

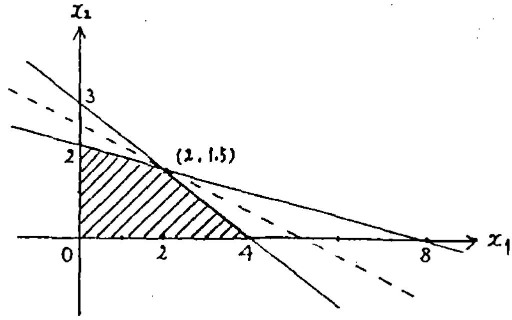

Fig.1 Optimal solutions of this distribution problem

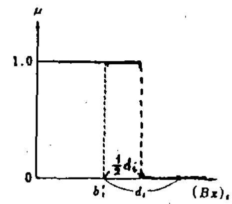

Fig.2 A half relaxing of upper-limits

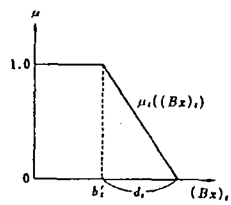

Fig.3 A linear-typed membership function