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Closed-Form Solution of ECA Target-Tracking Filter using Position and Velocity Measurement

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Abstract

Presented are closed-form expressions of the steady-state solution for the three-state exponentially correlated acceleration(ECA) target-tracking filter. The steady-state solution is derived based on Vaughan's approach for the case that the measurements of target position and velocity are available at discrete points in time. The solution for the ECA filter using only position measurements is obtained as a special case of the presented results.

I. Introduction

A realistic model for a maneuvering target has been proposed by Singer [1]. The Singer model assumes that that continuous time target motion may be represented with exponentially correlated acceleration(ECA). The discrete-time state equation of the target motion is simple, and it leads to a three-state Kalman filter solution for estimation and prediction of the target states.

The steady-state solution for the ECA filter has been extensively studied [2]-[5], [7], which provides *a priori* tracking performances and useful information for preliminary design. Fitzgerald presented the solution very efficiently with a careful parametrization for the case of using only position measurements in [2], and for the case of using position and velocity measurements in [3]. The steady-state solutions were generated by allowing the filters to run until the steady-state was reached. A closed-form solution for the ECA filter was obtained by Gupta [4], when only position measurements were available. The result is a generalization of the previous work of Gupta and Ahn [5], which is based on Vaughan's approach [6]. More recently, Beuzit [7] presented an alternative approach to obtain the closed-form solution based on the comparison between the Wiener and Kalman filtering.

On the other hand, Ramachandra [8] gave a closed-form solution for a constant-accelaration(CA) tracking filter with position measurements only. The tracking filter is derived under the assumption that the changes in the target acceleration, between two consecutive measurements, are a white noise process. The work is extended in [9] to the case that the position and velocity measurements are available.

In this paper we present closed-form expressions of the steady-state solution for the ECA tracking filter using the measurements of position and velocity. The steady-state solution is derived based on the Vaughan's results. The result of [4] is obtained as a special case of the presented expressions.

II. Equations of ECA Tracking Filter

The discrete-time model of ECA target motion is described by following equation:

$$x(k+1)=\Phi(T)x(k)+\upsilon(k) \tag{1}$$

where the dynamic state transtion matrix $\Phi(T)$ is given by

$$\Phi(T)=\begin{bmatrix}1 & \tau\theta & \tau^2 a_1\\ 0 & 1 & \tau(1-x)\\ 0 & 0 & x\end{bmatrix} \tag{2}$$

with $\theta=\frac{T}{\tau}$, $x=\exp(-\theta)$ and $a_1=\theta-1+x$.

Obviously,

$$\Phi^{-1}(T)=\begin{bmatrix}1 & -\tau\theta & -\tau^2 a_2\\ 0 & 1 & \tau(1-y)\\ 0 & 0 & y\end{bmatrix} \tag{3}$$

where $y=\exp(\theta)$, $a_2=\theta+1-y$. In eq.(1) $\upsilon(k)$ represents a stationary zero-mean white sequence with nonnegative definite covariance matrix Q given by

$$Q=E\,[\upsilon(k)\upsilon(k)^T]=\begin{bmatrix}q_{11} & q_{12} & q_{13}\\ q_{21} & q_{22} & q_{23}\\ q_{13} & q_{23} & q_{33}\end{bmatrix}. \tag{4}$$

The exact expression for Q is given in [1].

The position and velocity measurement, available every T second, are defined by

$$y(k) = Hx(k) + w(k) \tag{5}$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and $w(k)$ is a stationary zero-mean white sequence with positive definite covariance matrix R given by $diag\{\sigma_1^2, \sigma_2^2\}$. It is assumed that $w(k)$ is uncorrelated with $v(k)$.

III. Steady-state Solution for ECA Tracking Filter

For the ECA tracking filter, (Φ, H) and $(\Phi, Q^{\frac{1}{2}})$ are detectable and stabilizable, respectively. Thus, the steady-state prediction covariance matrix, denoted by P, exists and it is obtained by solving the discrete-time matrix Riccati equation

$$P = \Phi[P - PH^T(HPH^T + R)^{-1}HP]\Phi^T + Q. \tag{6}$$

Moreover, the steady-state Kalman gain K and the estimation covariance matrix, denoted by $\hat{P}$, are obtained, respectively, by computing

$$K = PH^T(HPH^T + R)^{-1} \tag{7}$$

and

$$\hat{P} = (1 - KH)P(1 - KH)^T + KRK^T. \tag{8}$$

The Vaughan's approach [6] to obtain the covariance matrix P is briefly outlined as follows.

1. Construct the Hamiltonian matrix of the Riccati eq.(6) such that

$$H_f = \begin{bmatrix} \Phi^{-T} & \Phi^{-T}H^TR^{-1}H \\ Q\Phi^{-T} & \Phi + Q\Phi^{-T}H^TR^{-1}H \end{bmatrix}.$$

2. Find the eigenvalues of H_f, $\lambda_i(H_f)$, satisfying $|\lambda_i(H_f)| > 1$, $i = 1, 2, 3$.
3. Find the eigenvector matrix W such that

$$WD = H_fW$$

with

$$D = \begin{vmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{vmatrix}, \quad \Lambda = diag\{\lambda_1, \lambda_2, \lambda_3\}.$$

4. The steady-state covariance matrix P is then given by

$$P = W_{21}W_{11}^{-1}$$

where W_{11}, W_{21} are partitioned matrices of W such that

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

Now, we describe the derivation of the covariance matrix P in detail. First, the Hamiltonian matrix is given by

$$H_f = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{\sigma_1^2} & 0 & 0 \\ \tau\theta & 1 & 0 & \frac{-\tau\theta}{\sigma_1^2} & \frac{1}{\sigma_1^2} & 0 \\ -\tau^2 a_2 & \tau(1-y) & y & \frac{-\tau^2 a_2}{\sigma_1^2} & \frac{\tau(1-y)}{\sigma_2^2} & 0 \\ U_1 & S_1 & yq_{13} & 1 + \frac{U_1}{\sigma_1^2} & \tau\theta + \frac{S_1}{\sigma_2^2} & \tau^2 a_1 \\ U_2 & S_2 & yq_{23} & \frac{U_2}{\sigma_1^2} & 1 + \frac{S_2}{\sigma_2^2} & \tau(1-x) \\ U_3 & S_3 & yq_{33} & \frac{U_3}{\sigma_1^2} & \frac{S_3}{\sigma_2^2} & x \end{bmatrix} \tag{9}$$

where

$$U_1 = q_{11} - \tau\theta q_{12} - \tau^2 a_2 q_{13}$$

$$U_2 = q_{12} - \tau\theta q_{22} - \tau^2 a_2 q_{23}$$

$$U_3 = q_{13} - \tau\theta q_{23} - \tau^2 a_2 q_{33}$$

$$S_1 = q_{12} + \tau(1-y)q_{13}$$

$$S_2 = q_{22} + \tau(1-y)q_{23}$$

$$S_3 = q_{23} + \tau(1-y)q_{33}.$$

The characteristic equation of the Hamiltonian is obtained by the determinent

$$|H_f - \lambda I| = 0 \tag{10}$$

and the eigenvectors are obtained by solving for x in

$$(H_f - \lambda I)x = 0. \tag{11}$$

By direct evaluation of eq.(10) we can determine the characteristic polynomial as

$$f(\lambda) = \lambda^6 + a\lambda^5 + b\lambda^4 + c\lambda^3 + b\lambda^2 + a\lambda + 1 = 0 \tag{12}$$

where

$$a = -4 - 2\cosh\theta - \frac{U_1}{\sigma_1^2} - \frac{S_2}{\sigma_2^2}$$

$$b = 7 + 8\cosh\theta + \frac{A_1}{\sigma_1^2} + \frac{A_2}{\sigma_2^2} + \frac{A_3}{\sigma_1^2\sigma_2^2}$$

$$c = -8 - 12\cosh\theta + \frac{B_1}{\sigma_1^2} + \frac{B_2}{\sigma_2^2} + \frac{B_3}{\sigma_1^2\sigma_2^2}$$

and

$$A_1 = 2(1 + \cosh\theta)U_1 - \tau\theta U_2 - \tau^2 a_1 U_3 + \tau\theta S_1 + \tau^2 a_2 y q_{13}$$

$$A_2 = 2(1 + \cosh\theta)S_2 - \tau(1-x)S_3 - \tau(1-y)yq_{23}$$

$$A_3 = U_1S_2 - U_2S_1$$

$$B_1 = -2(1+2\cosh\theta)U_1+\tau\theta(1+2\cosh\theta)U_2 + \tau^2[(2+y)a_1-\theta(1-x)]U_3-\tau\theta(1+2\cosh\theta)S_1 + \tau^2\theta^2 S_2+\tau^3\theta a_1 S_3+\tau^2[\theta(1-y)-(2+x)a_2]yq_{13} + \tau^3\theta a_2 y q_{23}+\tau^4 a_1 a_2 y q_{33}$$

$$B_2 = -2(1+2\cosh\theta)S_2+\tau(1-x)(2+y)S_3 + \tau(2+x)(1-y)yq_{23}-\tau^2(1-x)(1-y)yq_{33}$$

$$B_3 = -2\cosh\theta(U_1S_2-U_2S_1)+\tau(1-x)(U_1S_3-U_3S_1) + \tau^2 a_1(U_3S_2-U_2S_3)-\tau[\tau a_2 S_2+(1-y)U_2]yq_{13} + \tau[\tau a_2 S_1+(1-y)U_1]yq_{23}.$$

Let us define $X_i=\lambda_i+\lambda_i^{-1}$, $\quad i=1, 2, 3,$

so that

$$\lambda_i=\frac{X_i \pm \sqrt{X_i^2-4}}{2}, \qquad |\lambda_i|>1.$$

Factorizing the eigenvalue equation (12) such that

$$(\lambda-\lambda_1)(\lambda-\lambda_1^{-1})(\lambda-\lambda_2)(\lambda-\lambda_2^{-1})(\lambda-\lambda_3)(\lambda-\lambda_3^{-1})=0 \tag{13}$$

where λ_1, λ_2 and λ_3 are the eigenvalues outside the unit circle. Comparing (12) and (13) we get, after simplification,

$$X_1+X_2+X_3=\alpha$$
$$X_1X_2+X_2X_3+X_3X_1=\beta \tag{14}$$
$$X_1X_2X_3=\gamma$$

where

$\alpha=-a$

$\beta=b-3$

$\gamma=-c+2a.$

From (14) we can obtain

$$X_{1,2}=\frac{1}{2}[(\alpha-X_3)\pm\sqrt{(\alpha-X_3)^2-4X_3^{-1}\gamma}\ ] \tag{15}$$

and a cubic equation for X_3,

$$X_3^3-\alpha X_3^2+\beta X_3-\gamma=0. \tag{16}$$

The solutions of eq.(16) are obtained using the procedure detailed in [5]. Since the ECA model is of order 3, H_f is of order 6. If λ is an eigenvalue of H_f, then λ^{-1} is also an eigenvalue of H_f, and hence the eigenvalue problem is of third-order only. The eigenvector W_i corresponding to the eigenvalues λ_i are obtained by direct calculation as

$$W_i = \begin{bmatrix} 1 \\ w_{2i} \\ w_{3i} \\ w_{4i} \\ w_{5i} \\ w_{6i} \end{bmatrix} \tag{17}$$

where

$$w_{2i}=\frac{N_i}{D_i}$$

$$w_{3i}=\frac{1}{y-\lambda_i}\tau\lambda_i[\tau a_2-(1-y)(\tau\theta+\frac{N_i}{D_i})]$$

$$w_{4i}=(\lambda_i-1)\sigma_1^2$$

$$w_{5i}=(\tau\theta\lambda_i-(1-\lambda_i)\frac{N_i}{D_i})\sigma_2^2$$

$$w_{6i}=\frac{1}{x-\lambda_i}[-U_3\lambda_i-S_3\frac{N_i}{D_i}-w_{3i}yq_{33}-\frac{S_3}{\sigma_2^2}w_{5i}]$$

and

$$D_i=\frac{1}{\sigma_2^2}[\lambda_i\tau^2 y(1-y)[(1-x)q_{13}-\tau a_1 q_{23}] + (y-\lambda_i)[\tau(1-x)S_1-\tau^2 a_1 S_2]] - (1-\lambda_i)(y-\lambda_i)[\tau(1-x)(\tau\theta+\frac{S_1}{\sigma_2^2}) - \tau^2 a_1(1-\lambda_i+\frac{S_2}{\sigma_2^2})]$$

$$N_i=\frac{1}{\sigma_2^2}[-\tau(1-x)(y-\lambda_i)[U_1+(\lambda_i-1)(1-\lambda_i+\frac{U_1}{\sigma_1^2})\sigma_1^2] + \tau^2 a_1 U_2(y-\lambda_i)\lambda_i] - \frac{1}{\sigma_2^2}\tau^3 y\lambda_i[(1-x)q_{13}-\tau a_1 q_{23}][a_2-\theta(1-y)] - \tau\theta(y-\lambda_i)\lambda_i[\tau(1-x)(\tau\theta+\frac{S_1}{\sigma_2^2})-\tau^2 a_1(1-\lambda_i+\frac{S_2}{\sigma_2^2})]$$

The steady-state P matrix is now given by

$$P=W_{21}W_{11}^{-1} \tag{18}$$

where W_{11} and W_{21} are determined by the eigenvectors as

$$W_{11} = \begin{bmatrix} 1 & 1 & 1 \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}$$
$$\tag{19}$$
$$W_{21} = \begin{bmatrix} w_{41} & w_{42} & w_{43} \\ w_{51} & w_{52} & w_{53} \\ w_{61} & w_{62} & w_{63} \end{bmatrix}$$

The elements of the W_{11} and W_{21} are obtained by putting $i=1, 2, 3$ in eq.(17). Inverting W_{11} we obtain the expression

$$W_{11}^{-1} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (20)$$

where

$$a_{11} = \frac{1}{D}\Big[\ \tau^2(a_2-(1-y)\theta)\Big(\frac{N_2}{D_2}\frac{\lambda_3}{y-\lambda_3} - \frac{N_3}{D_3}\frac{\lambda_2}{y-\lambda_2}\Big) - \frac{N_2N_3}{D_2D_3}\tau(1-y)\Big(\frac{\lambda_3}{y-\lambda_3} - \frac{\lambda_2}{y-\lambda_2}\Big)\Big]$$

$$a_{12} = \frac{1}{D}\Big[\ \tau^2(a_2-(1-y)\theta)\Big(\frac{\lambda_2}{y-\lambda_2} - \frac{\lambda_3}{y-\lambda_3}\Big) - \tau(1-y)\Big(\frac{N_2}{D_2}\frac{\lambda_3}{y-\lambda_3} - \frac{N_3}{D_3}\frac{\lambda_2}{y-\lambda_2}\Big)\Big]$$

$$a_{13} = \frac{1}{D}\frac{D_2N_3-N_2D_3}{D_2D_3}$$

$$a_{21} = \frac{1}{D}\Big[\ \tau^2(a_2-(1-y)\theta)\Big(\frac{N_3}{D_3}\frac{\lambda_1}{y-\lambda_1} - \frac{N_1}{D_1}\frac{\lambda_3}{y-\lambda_3}\Big) - \frac{N_1N_3}{D_1D_3}\tau(1-y)\Big(\frac{\lambda_1}{y-\lambda_1} - \frac{\lambda_3}{y-\lambda_3}\Big)\Big]$$

$$a_{22} = \frac{1}{D}\Big[\ \tau^2(a_2-(1-y)\theta)\Big(\frac{\lambda_3}{y-\lambda_3} - \frac{\lambda_1}{y-\lambda_1}\Big) - \tau(1-y)\Big(\frac{N_3}{D_3}\frac{\lambda_3}{y-\lambda_3} - \frac{N_1}{D_1}\frac{\lambda_1}{y-\lambda_1}\Big)\Big]$$

$$a_{23} = \frac{1}{D}\frac{D_3N_1-N_3D_1}{D_1D_3}$$

$$a_{31} = \frac{1}{D}\Big[\ \tau^2(a_2-(1-y)\theta)\Big(\frac{N_1}{D_1}\frac{\lambda_2}{y-\lambda_2} - \frac{N_2}{D_2}\frac{\lambda_1}{y-\lambda_1}\Big) - \frac{N_1N_2}{D_1D_2}\tau(1-y)\Big(\frac{\lambda_2}{y-\lambda_2} - \frac{\lambda_1}{y-\lambda_1}\Big)\Big]$$

$$a_{32} = \frac{1}{D}\Big[\ \tau^2(a_2-(1-y)\theta)\Big(\frac{\lambda_1}{y-\lambda_1} - \frac{\lambda_2}{y-\lambda_2}\Big) - \tau(1-y)\Big(\frac{N_1}{D_1}\frac{\lambda_1}{y-\lambda_1} - \frac{N_2}{D_2}\frac{\lambda_2}{y-\lambda_2}\Big)\Big]$$

$$a_{33} = \frac{1}{D}\frac{D_1N_2-N_1D_2}{D_1D_2}$$

and

$$D = \tau^2[a_2-(1-y)\theta]\Big[\frac{N_1}{D_1}\frac{\lambda_2-\lambda_3}{(y-\lambda_2)(y-\lambda_3)} + \frac{N_2}{D_2}\frac{\lambda_3-\lambda_1}{(y-\lambda_3)(y-\lambda_1)} + \frac{D_3}{N_3}\frac{\lambda_1-\lambda_2}{(y-\lambda_1)(y-\lambda_2)}\Big] - \tau(1-y)\Big[\frac{N_1N_2}{D_1D_2}\frac{\lambda_2-\lambda_1}{(y-\lambda_2)(y-\lambda_1)} + \frac{N_2N_3}{D_2D_3}\frac{\lambda_3-\lambda_2}{(y-\lambda_3)(y-\lambda_2)} + \frac{N_1N_3}{D_1D_3}\frac{\lambda_1-\lambda_3}{(y-\lambda_1)(y-\lambda_3)}\Big]$$

The steady-state covariance $P = W_{21}W_{11}^{-1}$ then yields

$$\begin{aligned} P_{11} &= \sigma_1^2\sum_{i=1}^{3}(\lambda_i-1)a_{i1} \\ P_{12} &= \sigma_1^2\sum_{i=1}^{3}(\lambda_i-1)a_{i2} \\ P_{13} &= \sigma_1^2\sum_{i=1}^{3}(\lambda_i-1)a_{i3} \\ P_{22} &= \sigma_2^2\sum_{i=1}^{3}[\tau\theta\lambda_i-(1-\lambda_i)\frac{N_i}{D_i}]a_{i2} \\ P_{23} &= \sigma_2^2\sum_{i=1}^{3}[\tau\theta\lambda_i-(1-\lambda_i)\frac{N_i}{D_i}]a_{i3} \\ P_{33} &= \sum_{i=1}^{3}\frac{1}{x-\lambda_i}[-U_3\lambda_i-S_3\frac{N_i}{D_i}-w_{33}yq_{33}-\frac{S_3}{\sigma_2^2}w_{5i}\ ]a_{i3} \end{aligned} \qquad (21)$$

In the above, we derived equation eq.(21) for the steady-state prediction covariance matrix. Then, the steady-state Kalman gain K and the estimation covariance matrix $\hat{P}$ are determined as indicated in eqs.(7) and (8). Also, the steady-state smoothing covariance matrix [10, 11], denoted by P_s, can be obtained by solving a set of linear equations

$$P_s - AP_sA^T = \hat{P} - APA^T. \qquad (22)$$

where

$A = \hat{P}\Phi^T P^{-1}$.

The procedures to compute K, $\hat{P}$, and P_s are straighforward, and it is not detailed here.

We computed P using the derived expressions. Figs. 1, 2, and 3 present the results for the parametrization of $(\frac{\tau}{T}) = 10[0.56+3.4(\frac{\sigma_a T^2}{\sigma_1})^{-0.86}]^{\frac{1}{2}}$, where σ_a denotes the standard deviation of the exponentially correlated target acceleration. The same figures have been presented in [3]. The solution was inaccurate for $\sigma_2 T/\sigma_1 < 0.01$ and $\sigma_a T^2/\sigma_1 > 10$. The inaccuracy is due to the sensitivity of the solution of eq.(16), which depends on the parameters α, β, and γ.

IV. Discussion

In this paper we presented closed-form expressions of the steady-state solution for the ECA tracking filter using the measurements of position and velocity. Though it was not stated here, we could show that the expressions reduce to the result of [4] as $\sigma_2 \to \infty$. It is also expected that the results of [8, 9] are obtained as special cases of the presented expressions, which is currently under investigation.

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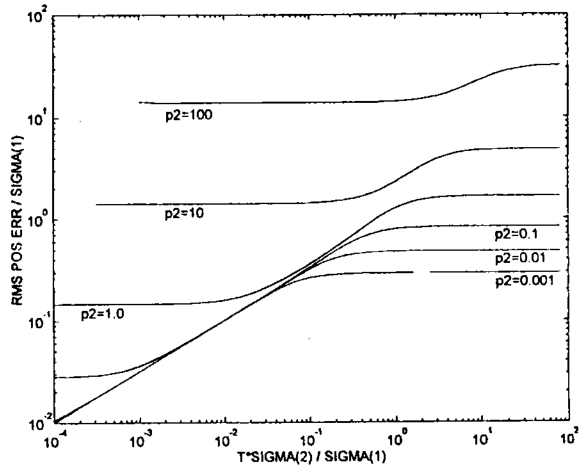

Fig. 1. Normalized rms position prediction errors.

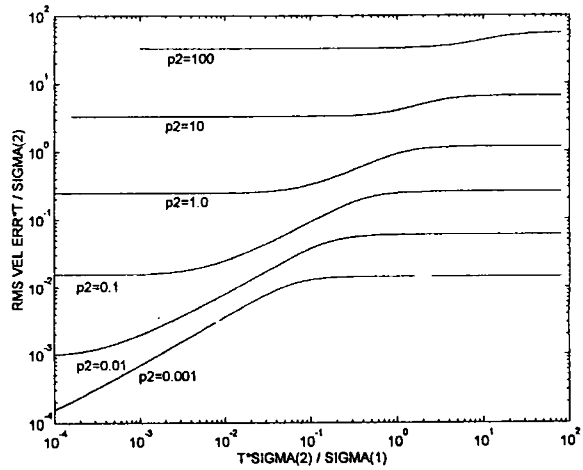

Fig. 2. Normalized rms velocity prediction errors.

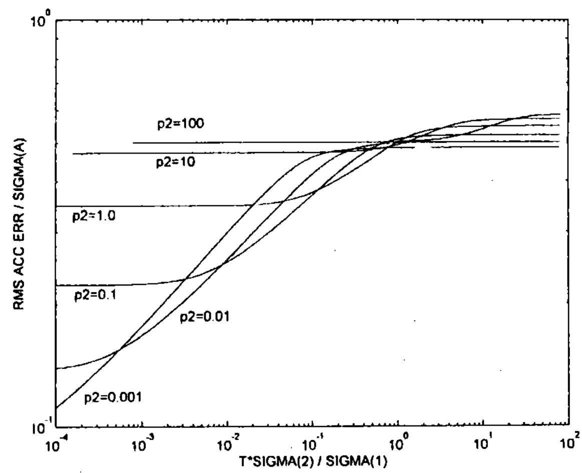

Fig. 3. Normalized rms acceleration prediction errors.