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Elastic Analysis of Plates Resting on Elastic Half-Space Considering the Local Segregation Between Plate and Foundation

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ABSTRACTS

It is one of classical problems in the elastic theory to analyze contact stresses between elastic bodies. Concrete pavements under traffic wheel loads can be considered as one of these typical problems. In this paper, an elastic plate resting on tensionless elastic half-space is analyzed by finite element method. The Boussinesq's solution of elastic half-space is used to evaluate the flexibility of foundation. One of the principal difficulties in solving the local seperation phenomena between plate and foundation is that the geometry of the system is unknown. To obtain the boundary of contact area, the flexibility matrix of foundation is modified after each cycle of analysis iteratively. Some numerical examples are presented by using these method.

INTRODUCTION

The response of elastic plates resting on a Boussinesq foundation has been studied by many invesgators under the assumption that the foundation reacts in compression as well as in tension. However, this assumption is questionable in many practical situations. The solution method required to determine the response of plates supported by tensionless foundation is complicated because the contact region is not known in advance.

In this paper, the plate problem is investigated by considering arbitrary types of loadings with no assumption on the shape of the contact region.

FINITE ELEMENT MODELING OF PLATES

We assume that the plate is idealized by the 8-node isoparametric plate elements[1]. In the formulation of the plate element, we use the assumptions adopted by Mindlin[3].

The stiffness relation of the plate can be written as the following equation.

$$\mathbf{P} = \mathbf{K}\,\underline{\delta} \tag{1}$$

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where **P** is the nodal load vector, **K** is the stiffness matrix and $\underline{\delta}$ is the nodal displacement vector of the plate. The degrees of freedom and the corresponding loads at a nodal point of a plate are shown in Figure 1.

MODELING OF ELASTIC HALF-SPACE

Boussinesq's Solution of Elastic Half-Space

The deflection of any point j due to a concentrated load P_i at point i on an elastic half-space is given by the Boussinesq's equation

$$w_{ij} = \frac{(1 - \nu_0^2)P_i}{\pi E_0 d_{ij}} \tag{2}$$

where ν_0 is the Poisson's ratio of the foundation, E_0 is the Young's modulus of the foundation and d_{ij} is the distance between points i and j. When distributed load acts on an arbitrarily shaped area the deflection of the point j can be obtained by integrating equation (2) over the area.

$$w_j = \frac{1 - \nu^2}{\pi E_0} \int_A \frac{q(x,y)dA}{\sqrt{(x_j - x)^2 + (y_j - y)^2}} \tag{3}$$

The Flexibility and Stiffness Matrices of the Elastic Half-Space

Consider a rectangular area e on a foundation. The region e is divided into 9 subregions around the Gauss points shown in Figure 2 and it is assumed that each contact pressure q_{Gi} is constant in that subregion. Thus the vertical displacement of point j is obtained by using equation (3)

$$w_j^{(e)} = \alpha\left[q_{G1} \int_{A_{G1}} \frac{dA_{G1}}{\sqrt{(x_j - x)^2 + (y_j - y)^2}} + \ . \ . \ . \ + q_{G9} \int_{A_{G9}} \frac{dA_{G9}}{\sqrt{(x_j - x)^2 + (y_j - y)^2}}\right] \tag{4}$$

or $w_j^{(e)} = \alpha \sum_{i=1}^{9} q_{Gi} I_{Gi}$ where $I_{Gi} = \int_{A_{Gi}} \frac{dA_{Gi}}{\sqrt{(x_j - x)^2 + (y_j - y)^2}}$ and $\alpha = \frac{1 - \nu^2}{\pi E_0}$

q_{Gi} is the contact pressure at the Gauss point i and A_{Gi} is the influence area where q_{Gi} is assumed to be constant. The integration I_{Gi} can be expressed as follows.

$$I_{Gi} = (y_{i2} - y_j) \ln \left| \frac{(x_{i2} - x_j) + \sqrt{(x_{i2} - x_j)^2 + (y_{i2} - y_j)^2}}{(x_{i1} - x_j) + \sqrt{(x_{i1} - x_j)^2 + (y_{i2} - y_j)^2}} \right| + (x_{i1} - x_j) \ln \left| \sqrt{1 + \frac{(y_{i2} - y_j)^2}{(x_{i1} - x_j)^2}} - \frac{(y_{i2} - y_j)}{(x_{i1} - x_j)} \right|$$

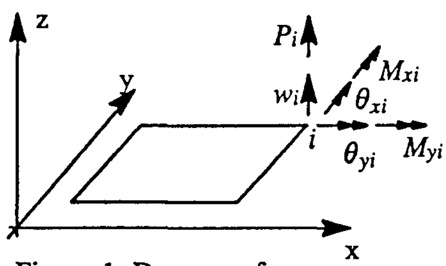

Figure 1. Degrees of freedom for a plate

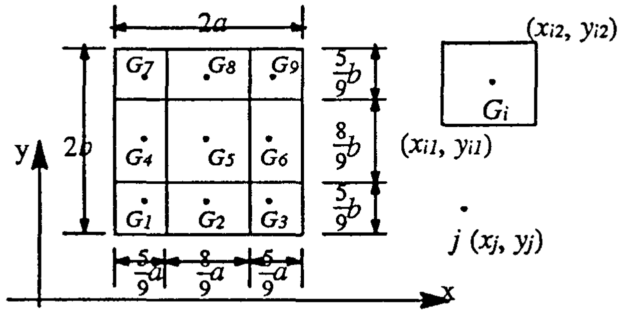

Figure 2. Subdivision of an element e

$$- (x_{i2} - x_j)\ln\left| \sqrt{1 + \frac{(y_{i2} - y_j)^2}{(x_{i2} - x_j)^2}} - \frac{(y_{i2} - y_j)}{(x_{i1} - x_j)} \right| - (y_{i1} - y_j)\ln\left| \frac{(x_{i2} - x_j) + \sqrt{(x_{i2} - x_j)^2 + (y_{i2} - y_j)^2}}{(x_{i1} - x_j) + \sqrt{(x_{i1} - x_j)^2 + (y_{i1} - y_j)^2}} \right|$$

$$- (x_{i1} - x_j)\ln\left| \sqrt{1 + \frac{(y_{i1} - y_j)^2}{(x_{i1} - x_j)^2}} - \frac{(y_{i1} - y_j)}{(x_{i1} - x_j)} \right| + (x_{i2} - x_j)\ln\left| \sqrt{1 + \frac{(y_{i1} - y_j)^2}{(x_{i2} - x_j)^2}} - \frac{(y_{i1} - y_j)}{(x_{i2} - x_j)} \right| \qquad (5)$$

At the Gauss point i within an element, the contact pressure q_{Gi} is interpolated with the nodal point values

$$q_{Gi} = \sum_{i=1}^{8} N_i q_i \quad = \mathbf{N}\ \mathbf{q}^{(e)} \qquad (6)$$

where $N_i = N_i(\xi_i, \eta_i)$ and is the matrix of two dimensional quadratic shape functions. Then we can write equation (4) by using the expressions (5) and (6).

$$w_j^{(e)} = \alpha[I_{G1}(x_j, y_j)\mathbf{N}(\xi_{G1}, \eta_{G1}) + ... + I_{G9}(x_j, y_j)\mathbf{N}(\xi_{G9}, \eta_{G9})]\ \mathbf{q}^{(e)} \qquad (7)$$

or

$$w_j^{(e)} = \mathbf{G}_j^{(e)}\mathbf{q}^{(e)} \qquad (8)$$

where

$$\mathbf{G}_j^{(e)} = [G_{j1}^{(e)}\ G_{j2}^{(e)}\ \cdot\ \cdot\ \cdot\ \cdot\ G_{j8}^{(e)}]$$

and

$$G_{ji}^{(e)} = \alpha \sum_{k=1}^{9} I_{Gk}(x_j, y_j)\ N_i(\xi_{Gk}, \eta_{Gk})] \qquad , i = 1, 2, \ldots, 8$$

Therefore, for the whole contact elements, the vertical deflection of the point j can be written as

$$w_j = \mathbf{G}_j\,\mathbf{q} \qquad (9)$$

where, $\mathbf{G}_j = \sum_{e=1}^{m} \mathbf{G}_j^{(e)} = [G_{j1}, G_{j2}, \cdot\cdot\cdot, G_{jn}]$ and $\mathbf{q} = [\,q_1\ q_2\ .\ .\ .\ .\ q_n]$

and m, n are the numbers of elements and nodal points, respectively. The summation means the assembly such as in the case of structural stiffness matrix. In a similar way, We can obtain the deflection of all points on the foundation.

$$\mathbf{w} = \mathbf{G}\,\mathbf{q} \qquad (10)$$

where $\mathbf{w} = [\,w_1\ w_2\ .\ .\ .\ .\ w_n]^T$ and $\mathbf{G} = \begin{bmatrix} G_{11} & G_{12} & \cdot & \cdot & G_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{n1} & G_{n2} & \cdot & \cdot & G_{nn} \end{bmatrix}$

G is the flexibility matrix of the foundation for the nodal contact pressure and is full and unsymmetric. This matrix has now to be combined with that of a plate subdivided into finite elements. Thus the contact pressure **q** must be transformed into the equivalent nodal forces **Q**. This can be done by the principle of virtual work, i.e., during any virtual displacements δw imposed on the system, the total work done by the contact pressure must be equal to total work done by the equivalent nodal forces. For one element,

$$[\delta \mathbf{w}^{(e)}]^T \mathbf{Q}^{(e)} = \int_{A^{(e)}} \delta w q(x,y)\, dA \tag{11}$$

The left side of equation (11) is replaced by the relation $q = \mathbf{N}\,\mathbf{q}^{(e)}$, $w = \mathbf{N}\,\mathbf{w}^{(e)}$ and arranged as

$$\mathbf{Q}^{(e)} = [\int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^T\mathbf{N}|\mathbf{J}|\, d\xi\, d\eta\,]\,\mathbf{q}^{(e)} \quad \text{or} \quad \mathbf{Q}^{(e)} = \mathbf{E}^{(e)}\mathbf{q}^{(e)} \tag{12}$$

where
$$\mathbf{E}^{(e)} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^T\mathbf{N}|\mathbf{J}|\, d\xi\, d\eta \tag{13}$$

Equation (12) can be extended to the total contact area by superposition.

$$\mathbf{Q} = \mathbf{E}\,\mathbf{q} \tag{14}$$

where
$$\mathbf{E} = \sum_{e=1}^{m} \mathbf{E}^{(e)} \tag{15}$$

From the equation (10)

$$\mathbf{q} = \mathbf{G}^{-1}\mathbf{w} \tag{16}$$

and by premultiplying $\mathbf{E}$ to both sides, we have

$$\mathbf{E}\,\mathbf{q} = \mathbf{E}\mathbf{G}^{-1}\mathbf{w} \quad \text{or} \quad \mathbf{Q} = \mathbf{K}_G\,\mathbf{w} \tag{17}$$

where $\mathbf{K}_G = \mathbf{E}\mathbf{G}^{-1}$ is the stiffness matrix of isotropic elastic half-space.

SOLUTION OF THE PLATE ON THE TENSIONLESS FOUNDATION

Construction of the combined equation

To obtain the combined load-deflection relationship of a plate and foundation, the difference of the number of degrees of freedom between a plate and foundation has to be considered. There are one vertical displacement and two rotations about the coordinate axes of the plane in the plate. But there is only one degree of freedom of vertical displacement in the half-space. Therefore the equation (17) needs to be extended to the same order of equation (1) [2]. The other difficulty comes from the fact that the flexibility matrix in the equation (10) is unsymmetric and full. In this study, the equation (1) is rearranged as

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ww} & \mathbf{K}_{w\theta} \\ \mathbf{K}_{\theta w} & \mathbf{K}_{\theta\theta} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \Theta \end{bmatrix} \tag{18}$$

where $\mathbf{P} = [\, P_1\ P_2 \cdots\cdots P_n\,]^T$, $\mathbf{M} = [\, M_{1x}\ M_{1y} \cdots\cdots M_{nx}\ M_{ny}]^T$

and
$$\Theta = [\theta_{1x}\ \theta_{1y}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \theta_{nx}\ \theta_{ny}\,]^T$$

and $\mathbf{K}$'s are corresponding matrices. Then by the matrix condensation, we have

$$\underline{\mathbf{P}} = \underline{\mathbf{K}}\,\mathbf{w} \qquad (19)$$

where $\underline{\mathbf{P}} = \mathbf{P} - \mathbf{K}_{w\theta}\,(\mathbf{K}_{\theta\theta})^{-1}\mathbf{M}$ and $\underline{\mathbf{K}} = \mathbf{K}_{ww} - \mathbf{K}_{w\theta}\,(\mathbf{K}_{\theta\theta})^{-1}\mathbf{K}_{\theta w}$

Thus the matrix $\underline{\mathbf{K}}$ in equation (19) has the same number of degrees of freedom with the matrix $\mathbf{K}_G$ in equation (17). The equilibrium condition of a plate is used to construct the combined equation of a plate and founadtion.

$$\underline{\mathbf{P}} - \mathbf{Q} = \underline{\mathbf{K}}\,\mathbf{w} \qquad (20)$$

from equation (17)

$$\underline{\mathbf{P}} = (\underline{\mathbf{K}} + \mathbf{K}_G)\,\mathbf{w} \qquad (21)$$

Local Separation between a Plate and a Foundation

There is no tensile stress between a plate and a foundation. The local separation phenomena take place when a foundation is hard and a plate on it is flexible . If the contact pressure of more than 4 nodes among the 8 nodes corresponding with the plate element are tensile stresses after one cycle of analysis has been completed, it is assumed that the area corresponding with that plate element doesn't contribute to the flexibility of the foundation and is excluded. Therefore the order of the flexibility matrix of the foundation needs to be adjusted. The iteration stops when the norm of displacements between the previous and current analysis results will becomes sufficiently small.

NUMERICAL EXAMPLES

Example 1

The rectangular plate subjected to uniformly distributed loading is analyzed, and its geometry, finite element discretisation and the material properties are shown in Figure 3.

The values of contact pressures along the center line (y = 0, nodes 161 - 181) are plotted in Figure 4 with the results of Ref.[4]. There are some differences between the results of this study and that in the Ref. [4] which used the modified Winkler model. It is hard to say which one describes the actual distribution of contact pressures more accurately. However, severe disturbance in the vicinity of the edges of a plate will be occurred in this case.

Example 2

A similar problem is analyzed with the thickness of 50 cm, fully distributed load of 12.26 kN/m^2 and partially distributed load of 300 kN/m^2(Figure 5). The displacements, bending moments M_x at the nearest Gauss points and the contact pressures along the center line are plotted in Figure 5, 6 and 7, respectively.

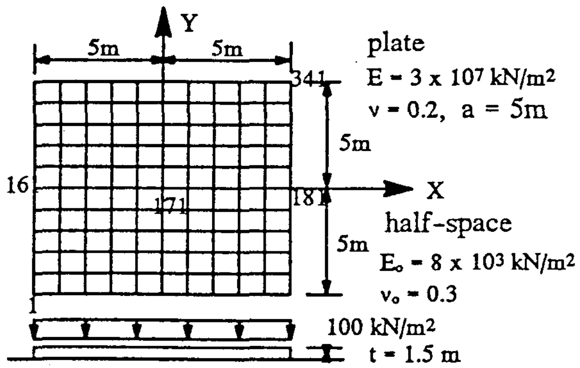

Figure 3. Example 1

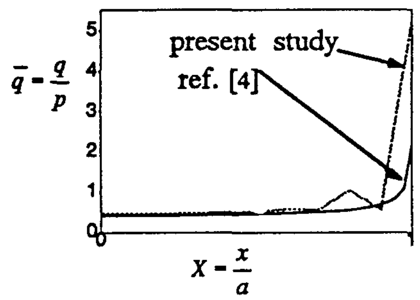

Figure 4. Contact Pressure distribution along the center line

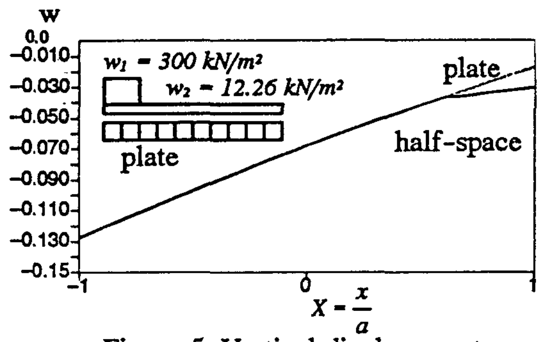

Figure 5. Vertical displacements along the center line

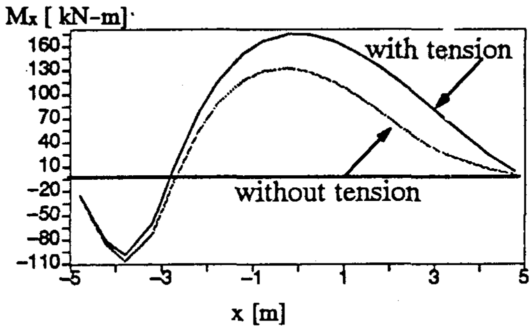

Figure 6. Bending Moment Diagram near the Center Line

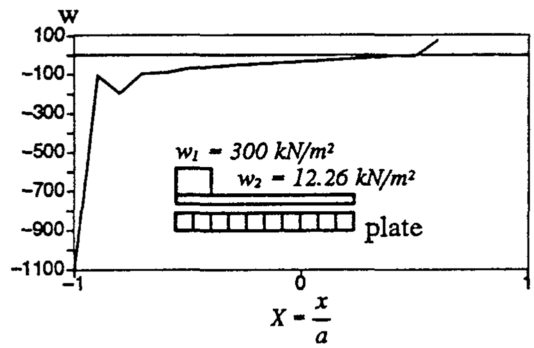

Figure 7. Contact Pressure Diagram

CONCLUSIONS

The contact region between a plate and a tensionless foundation has been determined successfully in a practical range. There are some tensile contact stresses at the boundaries but these can be eliminated or decreased by remeshing the plate to some extent. The bending moments of the plate are released by considering the tensionless foundation. The refined finite element meshes are required to obtain smooth contact pressure distribution near the edges of a plate.

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