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An Exponential Representation Form for Fuzzy Logic

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Abstract *By the exponential representation form (EF) for fuzzy logic, any fuzzy value a (in fuzzy valued logic or fuzzy linguistic valued logic[1]) can be represented as* B^c*, where B is called the truth base and* **c** *the confidence exponent. This paper will propose the basic concepts of this form and discuss its interesting properties. By using a different truth base, the exponential form can be used to represent the positive and the negative logic in fuzzy valued logic as well as in fuzzy linguistic valued logic. Some Simple application examples of EF for approximate reasoning are also illustrated in this paper.*

Keywords *Fuzzy Set Theory, Fuzzy Logic, Membership Functions, Approximate Reasoning.*

1. Introduction

Since fuzzy logic was first put forward, it has been used in a lot of real application areas. At the same time many researchers are also fascinated by the problems: 1) What are the basic properties of fuzzy logic? 2) Why can we process fuzzy logic problems by current computers that are built on binary logical basis? 3) Is it possible to find representation forms by which we can represent the operations on both binary and fuzzy logic in a common framework? The proposal of two kinds of fuzzy truth values (*truth I* and *truth II*)[1]; the discussions on *fuzzy positive logic* and *fuzzy negative logic*[2] and the proposal of *fuzzy confidence*[3] can be looked as one part of the attempts for exploring the properties of fuzzy logic. In continuance of the above research, a new representation called the *exponential form* (EF for short) on fuzzy logic is proposed in this paper.

The basic definitions of EF are given in fuzzy valued logic first. For any fuzzy value $a \in [0,1]$, we can represent it by an EF B^c, where $B \in [0,1]$ is called the *truth base* and $c \in (-\infty, \infty)$ is called the *confidence exponent*. Suppose 0 means false, 1 means true and 0.5 means unknown in a fuzzy logical system; then when $B > 0.5$, the EF represents the so-called *fuzzy positive logic*, and when $B < 0.5$, the EF represents the so-called *fuzzy negative logic*. When $B = 0.5$, the EF is *meaningless* (for reasons illustrated below). Further, when $B \neq 0$ and $B \neq 1$, the EF can also be looked as a representation of the fuzziness of fuzzy, or called *second order of fuzzy*. It is important that a fuzzy value represented by EF can be transformed equivalently from one truth base to another truth base. This means that fuzzy operations can be easily executed in various fuzzy values by changing the representation of their EF to a same truth base. By extending the truth base and the confidence exponent from a single value in the interval [0, 1] to a fuzzy set in the space $[0, 1]^2$, EF can also represent fuzzy linguistic valued logic.

In section 2, we give the basic definitions on two fuzzy truths. The basic definitions and properties of EF are given in section 3 and section 4, respectively. The application of EF for approximate reasoning is discussed in section 5 by simple examples. The section 6 is conclusion.

2. Definitions on Two Fuzzy Truths

Fuzzy valued logic can be defined as an algebraic system < { [0, 1], F, T, unknown, undefined }, ∧, ∨, ¬ >. Where in the truth set { [0, 1], F, T, unknown, undefined }, the closed interval [0, 1] is a set of truth values, the $F \subset [0, 1]$ is a set of values on degree of false, the $T \subset [0, 1]$ is a set of values on degree of true, the $\underline{unknown} \subset [0, 1]$ is a special set represented by only one point where both F and T are unknown, and the set $\underline{undefined} \not\subset [0, 1]$. The logical operations AND(∧), OR(∨) and NOT(¬) in the algebraic system are defined as follows:

$$A \wedge B = \min(A, B), \quad (2.1)$$

$$A \vee B = \max(A, B), \quad (2.2)$$

$$\neg A \text{ (or not (A))} = 1 - A, \quad A, B \in [0, 1]. \quad (2.3)$$

Definition 2.1: Let the kind of truth be *truth I* in fuzzy valued logic if and only if in the truth set, F = undefined, $T \in (0, 1]$, and unknown = 0. (Fig. 2.1)

Definition 2.2: Let the kind of truth be *truth II* in fuzzy valued logic if and only if in the truth set, $F \in [0, 0.5)$, $T \in (0.5, 1]$, and unknown = 0.5. (Fig. 2.2)

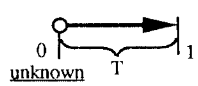

Fig. 2.1. The *truth I*

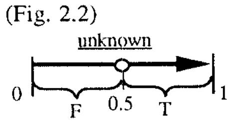

Fig. 2.2. The *truth II*

According to the above definition, under *truth II,* if a be an element of V = [0, 1], then a = 0.5 is said to be *unknown.* In other words, that is, in the set of truth value [0, 1], it is considered that 0 and 1 have different definite information and the ambiguity reaches its maximum at 0.5. But, under *truth I,* it is considered that 1 has definite information and the ambiguity reaches its maximum at 0.

Definition 2.3: Let a be an element of V = [0,1]. Then, the *confidence* of a is as follows:

$$C(a) = a \qquad (2.4)$$

$C(a) \in [0, 1]$ and $a \in [0, 1]$ under *truth I*;

$$C(a) = (a - 0.5) \times 2 \qquad (2.5)$$

$C(a) \in [-1, 1]$ and $a \in [0, 1]$ under *truth II*;

and the absolute value $| C(a) |$ will designate the non-ambiguity of a.

Example 2.1: If under *truth II,* a = 0.8, b = 0.2 then C(a) = (0.8 – 0.5) × 2 = 0.6 and C(b) = (0.2 – 0.5) × 2 = –0.6, respectively. Therefore, $| C(a) | = | C(b) | = 0.6$, which means they have same absolute values of confidence and ambiguity in the T part and the F part respectively.

A truth value which is a point in [0, 1], e.g. T(A) = 0.8, will be referred to as a *numerical* truth value in fuzzy valued logic. The numerical truth values play the role of the values of the base variable for the linguistic variable *Truth*. The linguistic values of the *Truth* will be referred to as *linguistic truth values*, and by which so-called *fuzzy linguistic valued logic* can be defined.

Fuzzy linguistic valued logic can be defined as an algebraic system < { T1, T2, **F**, **T**, **unknown**}, ∧, ∨, ¬ >. Where T1 and T2 are the truth sets in the fuzzy valued logic, and they are the universe of discourse of truth and the membership degree of the truth, respectively. Any linguistic truth value A ∈ { **F**, **T** } is a fuzzy subset $A = \{ \mu_A(\tau)/\tau \mid \tau \in T1, \mu_A(\tau) \in T2 \}$; the **F** means a set of linguistic truth values on false; the **T** means a set of linguistic values on true; the **unknown** is a set defined by $\int 0/\tau$. The logical operations **AND**(∧), **OR**(∨) and **NOT**(¬) are defined as follows:

$$A \wedge B = \int \min(\mu_A(\tau), \mu_B(\tau))/\tau, \qquad (2.6)$$

$$A \vee B = \int \max(\mu_A(\tau), \mu_B(\tau))/\tau, \qquad (2.7)$$

$$\neg A \ \ (\text{or } \mathbf{not}(A)) = \int (1 - \mu_A(\tau))/\tau, \qquad (2.8)$$

where, $\int$ means union over *T*, and $A, B \subset \{ \mathbf{F}, \mathbf{T} \}, \tau \in [0, 1] \in T1; \mu_A(\tau), \mu_B(\tau) \in [0, 1] \in T2$.

Definition 2.4: Let the type of truth be *truth I-I* in fuzzy linguistic valued logic if and only if in the truth set , T1 and T2 are all *truth I* in fuzzy valued logic.

Dependent on Definition 2.4, a popular kind of *truth I-I* of fuzzy linguistic valued logic systems may be defined as follows (Fig. 2.4):

$$\text{true} = \int \mu_{true}(\tau)/\tau,$$

$$\text{for } \mu_{true}(\tau) = \tau, \tau \in [0, 1] \qquad (2.9)$$

Definition 2.5: Let the kind of truth be *truth II-I* in fuzzy linguistic valued logic if and only if in the truth set, T1 is *truth II* and T2 is *truth I* in fuzzy valued logic.

Dependent on Definition 2.5, another popular kind of *truth II-I* of fuzzy linguistic valued logic systems may be defined as follows (Fig. 2.5):

$$\text{true} = \int \mu_{true}(\tau)/\tau, \qquad (2.10)$$

$$\text{for } \mu_{true}(\tau) = 0, \tau \in [0, 0.5]$$

$$\mu_{true}(\tau) = (\tau - 0.5) \times 2, \tau \in [0.5, 1]$$

and

$$\text{false} = \int \mu_{false}(\tau)/\tau, \qquad (2.11)$$

$$\text{for } \mu_{false}(\tau) = (0.5 - \tau) \times 2, \tau \in [0, 0.5]$$

$$\mu_{false}(\tau) = 0, \tau \in [0.5, 1]$$

In real applications, the types of linguistic *truth I-I* and *II-I* are often to be used.

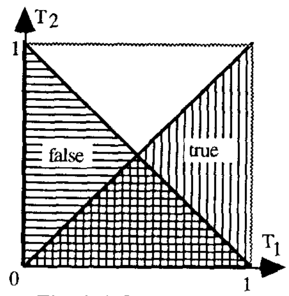

Fig. 2.4. The *truth I-I*

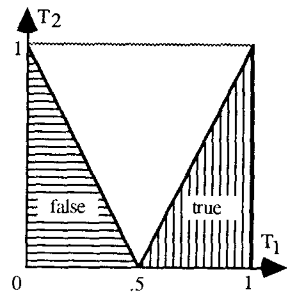

Fig. 2.5. The *truth II-I*

In this paper, we mainly discuss EF on fuzzy valued logic and leave to other papers the detailed discussion of EF on the fuzzy linguistic valued logic.

3. Basic Definitions of EF on Fuzzy Valued Logic

Definition 3.1(Basic Expression):

Let a ∈ [0, 1] be a true value in fuzzy valued logic, then a is equivalent to an *exponential form* B^c if and only if

$$a = (B - \mathbf{U}) \times \mathbf{c} + \mathbf{U}. \qquad (3.1)$$

where, B ∈ [0, 1] is called *fuzzy truth base*, $\mathbf{c} \in (-\infty, \infty)$ is called *confidence exponent*, **U** = <u>unknown</u> is called *unknown* or *meaningless point* for inference, which is equal to 0 (*truth I*) or 0.5 (*truth II*).

Obviously, as the special cases to let B be 1 in *truth I*, the following formula holds:

$$a = 1^c = \mathbf{c} \qquad (3.2)$$

and where $\mathbf{c} \in [0, 1]$ is always satisfied. Moreover, if **c** = 0, then a ≡ 0 for any fuzzy truth base, that is,

$$B^0 \equiv 0 \qquad B \in [0, 1] \qquad (3.3)$$

which means in value 0, the ambiguity reaches maximum and the confidence reaches minimum. On the other hand, if let B = 0 in *truth I,* then

$$0^c \equiv 0 \qquad \mathbf{c} \in [0, \infty) \qquad (3.4)$$

So we say B = 0 or **c** = 0 in *truth I* will cause meaningless for logical inference.

Similarly, to let B be 1 or 0 in *truth II* respectively, the following formulas hold:

$$a = 1^{c} = c \times 0.5 + 0.5 \quad (3.5)$$

$$a = 0^{c} = -c \times 0.5 + 0.5 \quad (3.6)$$

and where $c \in [-1, 1]$ is always satisfied. Moreover, if $c = 0$, then $a \equiv 0.5$ for any fuzzy truth base, that is,

$$B^{0} \equiv 0.5 \qquad B \in [0, 1] \quad (3.7)$$

which means at the value 0.5, the ambiguity reaches maximum and the confidence reaches minimum. If let $B > 0.5$, then B^c is said to be the representation of *fuzzy positive logic* of a. If let $B < 0.5$, then B^c is said to be the representation of *fuzzy negative logic* of a. And if let $B = 0.5$, then

$$0.5^{c} \equiv 0.5 \qquad c \in (-\infty, \infty) \quad (3.8)$$

So we say $B = 0.5$ or $c = 0$ in *truth II* will cause meaningless for logical inference.

Definition 3.2(Fuzzy True and Fuzzy False in *truth II*):

Let $a \in [0, 1]$ be a true value in fuzzy valued logic (*truth II*) and

$$a = B^{c}, \quad (3.9)$$

then a is said to be *fuzzy true* in fuzzy positive logic (fuzzy negative logic) when $B = 1 (B = 0)$ and $c \in (0, 1]$, or *fuzzy false* in fuzzy positive logic (fuzzy negative logic) when $B = 1$ $(B = 0)$ and $c \in [-1, 0)$.

Definition 3.3 (Full Confidence, Incomplete Confidence and Super Confidence):

Suppose a fuzzy value a (*truth I* or *truth II*) represented by an exponential form $B^c = a$. If $|c| = 1$ then a is said to be *full confidence* on the truth base B; if $0 < |c| < 1$ then a is said to be *incomplete confidence* on the truth base B; and if $1 < |c| < \infty$ then a is said to be *super confidence* on the truth base B, where $B \in [0, 1]$.

4. Properties of Exponential Form on Fuzzy Valued Logic

Property 4.1(Logical operations):

For $B \in [0, 1]$, $c_i \in [0, \infty)$ $(i = 1, 2, ..., n)$ in *truth I*, the following formulas for logical operations hold:

$$B^{c_1} \vee ... \vee B^{c_i} \vee ... \vee B^{c_n} = B^{\vee(c_1,c_2,...,c_n)} \quad (4.1)$$

$$B^{c_1} \vee ... \vee B^{c_i} \vee ... \vee B^{c_n} = B^{\wedge(c_1,c_2,...,c_n)} \quad (4.2)$$

For $B \in [0, 1]$, $c_i \in (-\infty, \infty)$ $(i = 1, 2, ..., n)$ in *truth II*, the following formulas for logical operations hold:

$$B^{c_1} \vee ... \vee B^{c_i} \vee ... \vee B^{c_n} = B^{\vee(c_1,c_2,...,c_n)} \quad (4.3)$$

$$B^{c_1} \vee ... \vee B^{c_i} \vee ... \vee B^{c_n} = B^{\wedge(c_1,c_2,...,c_n)} \quad (4.4)$$

$$\neg(B^{c_i}) = B^{-c_i} = (1 - B)^{c_i} \quad (4.5)$$

where, $\vee$, $\wedge$ and $\neg$ mean OR(max), AND(min) and NOT(1−) operations, respectively.

Property 4.2(Equality of Confidence 1):

For any value $a \in [0, 1]$ in *truth I* or *truth II*, let the truth base $B = a$, then the following relation is always satisfied:

$$B^{1} = a^{1} = a \quad (4.6)$$

This property shows that any value $a \in [0, 1]$ has confidence **1** on the truth base as itself.

Property 4.3(Base Changing Formula):

The exponential form of a fuzzy value a on truth base B_1 can be changed to the form on another truth base B_2 by the following formula:

$$a = B_1^{c_1} = B_2^{c_2} \quad (4.7)$$

if

$$c_2 = c_1 \times (B_1 - U) \div (B_2 - U) \quad (4.8)$$

where U is the unknown point.

In *truth I*, $U = 0$ and in *truth II*, $U = 0.5$ are used. But U can also be any point in the interval [0,1]. That is, many different types of *truth* can be used in real applications.

Property 4.4 (Simplification of High-order Fuzziness):

For any high-order exponential form, there is an equivalent relation:

$$(B^{c_1})^{c_2} = B^{c_1 \times c_2} \quad (4.9)$$

By the above properties, we can change a fuzzy value from one truth base to another, so that the logical operations can be executed among the values given on different truth bases. Also the fuzzy positive logic and the fuzzy negative logic on *truth II* can be linked together by these properties, e.g.:

$$1^{c} \equiv 0^{-c} \qquad \text{and} \qquad 1^{-c} \equiv 0^{c} \quad (4.10)$$

5. The Applications of Exponential Form

The exponential form on fuzzy logic has provided a possibility to deal with high-order fuzziness in reasoning based on its properties. When it is used for approximate reasoning, all truth values are changed to a common truth base first, and then an approximate reasoning will be done by their confidence exponents on the truth base. Finally, the result of the inference is changed back to the original truth base or any convenient truth base.

5.1 Approximate Reasoning in Fuzzy Valued Logic

Example 5.1: Suppose a fuzzy rule $P \rightarrow Q$ and a fact P', an approximation of P, are given on *truth I* of fuzzy valued logic. Where, P = "a is 0.8 true", Q = "b is 0.9 true", T(P) = 0.9 and T(Q) = 0.8, P' = "a is 0.7 true" and T(P') = 0.9.

(1) By EF, we can represent P, Q and P' as $0.8^{0.9}$, $0.9^{0.8}$, $0.7^{0.9}$, respectively.

(2) Using Property 4.3, we can change them to a common truth base B = 1:

$0.8^{0.9} = 1^{0.72}$ $\qquad 0.9^{0.8} = 1^{0.72}$ $\qquad 0.7^{0.9} = 1^{0.63}$

respectively.

(3) An approximate reasoning is done on confidence exponent 0.72, 0.72 and 0.63. In other words, instead of P, Q , P', we use P* = 0.72, Q* = 0.72 and P** = 0.63 to do approximate reasoning. Using the linear revising method of revision principle[4,6], we can get Q** = 0.63.

(4) Using Property 4.3, we can have

$$1^{0.63} = 0.9^{0.7}$$

Namely, " 'b is 0.9 true' is 0.7 true" is approximately deduced.

5.2 Approximate Reasoning in Fuzzy Linguistic Valued Logic

The basic definitions of EF are given on fuzzy valued logic, where the truth base $B \in [0, 1]$ and the confidence exponent $\mathbf{c} \in (-\infty, \infty)$. By extending the truth base and the confidence exponent from a single value to the membership function of a fuzzy set (linguistic truth value), it is also possible to apply EF in *fuzzy linguistic valued logic.*

Example 5.2: Suppose a fuzzy rule $P \rightarrow Q$ and P', an approximation of P, are given on fuzzy linguistic valued logic in *truth I-I*, and P = *true*, T(P) = *very true*, Q = *very true*, T(Q) = *true*, P' = *more or less true*, T(P') = *very true*, where the fuzzy sets *true*, *very true* and *more or less true* are defined as:

true = $\int \tau/\tau$,

very true = $\int \tau^2/\tau$,

more or less true = $\int \tau^{0.5}/\tau$, $\tau \in [0, 1]$.

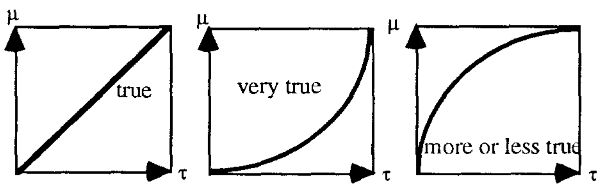

Fig. 5.1 *true*, *very true* and *more or less true* in *truth I-I*

(1) Representing P, Q, P' with their truth values by an extending EF:

$$P = B_P(\tau)^{c_P(\tau)} \quad (5.1)$$

$$Q = B_Q(\tau)^{c_Q(\tau)} \quad (5.2)$$

$$P' = B_{P'}(\tau)^{c_{P'}(\tau)} \quad (5.3)$$

where, $B_P(\tau)$, $B_Q(\tau)$, $B_{P'}(\tau)$ are fuzzy sets *true*, *very true*, *more or less true*, respectively, and $c_P(\tau)$, $c_Q(\tau)$, $c_{P'}(\tau)$ are fuzzy sets *very true*, *true*, *very true*, respectively.

(2) Changing P, Q, P' to a *common truth base* $\mathbf{B}(\tau)$. Suppose the common truth base

$$\mathbf{B}(\tau) = \int \tau/\tau \quad (5.4)$$

and the unknown

$$\mathbf{U}(\tau) = \int 0/\tau \quad (5.5)$$

are applied, then by Property 4.3 we have:

$$P = \mathbf{B}(\tau)^{\mathbf{c'}_P(\tau)} \quad (5.6)$$

$$Q = \mathbf{B}(\tau)^{\mathbf{c'}_Q(\tau)} \quad (5.7)$$

$$P' = \mathbf{B}(\tau)^{\mathbf{c'}_{P'}(\tau)} \quad (5.8)$$

and

$$\mathbf{c'}_P(\tau) = \int(\tau^2 \times \tau) \div \tau / \tau = \int \tau^2/\tau, \quad (5.9)$$

$$\mathbf{c'}_Q(\tau) = \int(\tau \times \tau^2) \div \tau / \tau = \int \tau^2/\tau, \quad (5.10)$$

$$\mathbf{c'}_{P'}(\tau) = \int(\tau^2 \times \tau^{0.5}) \div \tau / \tau = \int \tau^{1.5}/\tau. \quad (5.11)$$

(3) Approximate reasoning is done on confidence exponent functions $\mathbf{c'}_P(\tau)$, $\mathbf{c'}_Q(\tau)$ and $\mathbf{c'}_{P'}(\tau)$. By using the formulas of the revision principle[4,5,6], we can get:

$$\mathbf{c'}_{Q'}(\tau) = \mathbf{c'}_Q(\tau) + (\mathbf{c'}_{P'}(\tau) - \mathbf{c'}_P(\tau))$$
$$= \int(\tau^2 + (\tau^{1.5} - \tau^2)) / \tau = \int \tau^{1.5}/\tau. \quad (5.12)$$

(4) Using Property 4.3 and the definitions of *true*, *very true*, *more or less true*, we can get an equivalent of $\mathbf{B}(\tau)^{\mathbf{c'}_{Q'}(\tau)}$ on base $B_Q(\tau)$ by the equation:

$$c_{Q'}(\tau) = \mathbf{c'}_{Q'}(\tau) \times \mathbf{B}(\tau) \div B_Q(\tau)$$
$$= \int(\tau^{1.5} \times \tau \div \tau^2)) / \tau = \int \tau^{0.5}/\tau. \quad (5.13)$$

and

$$Q' = \mathbf{B}(\tau)^{\mathbf{c'}_{Q'}(\tau)} = B_Q(\tau)^{c_{Q'}(\tau)} \quad (5.14)$$

where $B_Q(\tau)$ are fuzzy set *very true* and

$$c_{Q'}(\tau) = \int \tau^{0.5}/\tau = \textit{more or less true}.$$

That is, "Q' = *very true* and T(Q') = *more or less true*" is approximately deduced. Obviously, this is a reasonable conclusion.

6. Conclusion

We proposed one kind of representations called exponential form(EF) for fuzzy logic. The possible applications on approximate reasoning have been illustrated by examples. The basic definitions and some important properties of EF have been given on fuzzy valued logic, but by extending the truth base and the exponent confidence to fuzzy set, it is also possible to apply EF on fuzzy linguistic valued logic and other kinds of fuzzy logic.

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