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ON STABILITY ANALYSIS OF NONLINEAR PLANTS WITH FUZZY LOGIC CONTROLLERS

K.-H. Cho, C.-W. Kim, and J.-T. Lim

Department of Electrical Engineering
Korea Advanced Institute of Science and Technology
373-1, Kuseong-dong, Yusung-gu, Taejon, 305-701, Korea

Abstract

In this paper, the absolute stability criterion of nonlinear plants with sector bounded nonlinear feedback is derived. The result obtained is useful for applications, in particular, stability analysis and design of fuzzy logic controllers.

I. INTRODUCTION

The problem of absolute stability is as follows: given a linear time invariant stable plant controlled by an unknown nonlinear, memoryless feedback $\phi(y)$ belonging to a sector on $(y, \phi(y))$-plane, find the largest sector which ensures the global asymptotic stability of the closed loop system. The solutions obtained range from sufficient and necessary conditions [1]-[4] to computer oriented techniques [5]. However, the literature does not seem to offer any result concerning nonlinear plants with a sector bounded nonlinear memoryless feedback. Such results are of certain significance both by themselves and, more importantly, in connection with fuzzy logic controllers. Fuzzy logic controllers have proved to be very successful in a number of practical applications [6]-[10]. Though fuzzy logic controllers may provide superior results, there is no systematic ways for their stability analysis. The only exception is the application of the classical absolute stability theory [11]-[13]. In this work, the authors assumed that, for the purpose of analysis, the model of the plant is known and, moreover, is linear and time invariant. The controller is assumed to be fuzzy and is represented as a sector bounded nonlinearity. Then the absolute stability theory is applied in order to analyze the closed loop stability or robustness properties, using the gain-phase margin philosophy. The purpose of this paper is two-fold. First, we develop two stability criterions by a simple extention of classical absolute stability theory for nonlinear plants described in Section II, III below. Second, for each case, following the approach of [11]-[13], we apply the criterions obtained to stability analysis of nonlinear plants with fuzzy logic controllers. Finally, we formulate the conclusions in Section IV.

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II. INFINITE SECTOR BOUND CRITERION

1. Stability Criterions

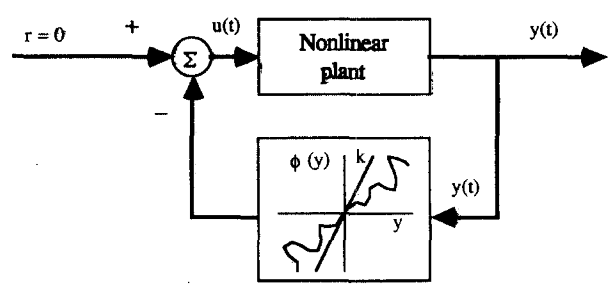

Fig.1 Nonlinear system with sector bounded feedback

For the above system, consider the nolinear plant defined as follows:

$$\dot{x} = f(x) + g(x)u \quad (1)$$

$$y = h(x) \quad (2)$$

where $x \in R^n$ is the state, $y \in R$ is the output, $u \in R$ is the control input and $f(\cdot), g(\cdot) \in R^n$, $h(\cdot) \in R$ are vector and scalar functions, respectively, that are continuously differentiable for all $x \in R^n$, and $f(0) = 0, h(0) = 0$. The control input u is chosen in the feedback manner as follows:

$$u = -\phi(y) \quad (3)$$

where $\phi(\cdot)$ is a continuous function belonging to a sector $[0, k]$, *i.e.* $0 \leq \phi(y)y \leq ky^2, \forall y \in R$. Thus, the closed loop system is

$$\dot{x} = f(x) - g(x)\phi(h(x)). \quad (4)$$

Theorem 1 Given the system defined by $(1) - (3)$, assume that there exists a positive definite matrix P such that the matrix $R(x) = -(A^T(x)P + PA(x))$ is positive definite in some neighborhood Ω of the origin of R^n where $A(x) = \frac{\partial f(x)}{\partial x}$. Then the equilibrium point at the origin is asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[0, \infty)$ if there exists $\beta \geq 0$ such that for all $x \neq 0$ in Ω,

$$\beta(\frac{\partial h(x)}{\partial x})g(x) - v^T(x)R^{-1}(x)v(x) = 0,$$

where $v(x) = PA(x)g(x) - \frac{1}{2}\beta(\frac{\partial h(x)}{\partial x})^T$,
and $w(x) = f(x)R(x) + v(x)\phi(y) = 0$ only for $x = 0$.
If $\Omega = R^n$ and $f^T(x)f(x)$ is radially unbounded, then the equilibrium point at the origin is globally asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[\ 0,\infty\ )$.

Proof of Theorem 1 : Consider

$$\begin{aligned}
V &= f^T(x)Pf(x) + \beta\int_0^y \phi(\sigma)d\sigma \\
\text{Then } \dot{V} &= [A(f+gu)^T]Pf + f^TP[A(f+gu)] - \beta u(\frac{\partial h}{\partial x})(f+gu) \\
&= f^T(A^TP + PA)f + 2f^TPAgu - \beta f^T(\frac{\partial h}{\partial x})^T - \beta(\frac{\partial h}{\partial x})gu^2 \\
&= -[f^TRf - 2f^T(PAg - \frac{1}{2}\beta(\frac{\partial h}{\partial x})^T)u] - \beta(\frac{\partial h}{\partial x})gu^2 \\
&= -[f^TRf - 2f^Tvu] - \beta(\frac{\partial h}{\partial x})gu^2 \ (v = PAg - \frac{1}{2}\beta(\frac{\partial h}{\partial x})^T) \\
&= -[(f - R^{-1}vu)^TR(f - R^{-1}vu)] - [\beta(\frac{\partial h}{\partial x})g - v^TR^{-1}v]u^2 \\
\dot{V} & \text{ is negative definite if } \beta(\frac{\partial h}{\partial x})g - v^TR^{-1}v = 0 \\
\text{and } & f - R^{-1}vu = 0 \text{ only for } x = 0 \text{ Q.E.D.}
\end{aligned}$$

Let's consider the following decoupled system:

$$\dot{x} = F(x)x + g(x)u \quad (5)$$

$$y = H(x)x \quad (6)$$

$$u = -\phi(y) \quad (7)$$

Then, we show the following corollary.

Corollary 1 Given the system defined by $(5) - (7)$, assume that there exists a positive definte symmetric matrix P such that the matrix $Q(x) = -(F^T(x)P + PF(x))$ is positive definite in some neighborhood Ω of the origin of R^n. Then the equilibrium point at the origin is asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[0, \infty)$ if any of the following two conditions is met. If $\Omega = R^n$, then the equilibrium point at the origin is globally asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[\ 0, \infty\ )$.

1. There exists $\beta > 0$ such that $\beta(\frac{\partial y}{\partial x})^Tg(x) - v^T(x)Q^{-1}(x)v(x) = 0$ and $x^Tv(x)y \geq 0$, where $v(x) = Pg(x) - \frac{\beta}{2}F^T(x)(\frac{\partial y}{\partial x})^T$.
2. If $\beta = 0$, then $x^TPg(x)y \geq 0$.

If we take $V = x^TPx + \beta\int_0^y \phi(\sigma)d\sigma$, the proof of the above is analogous to the theorem 1, so it is omitted here.

2. Stability Analysis with Fuzzy Logic Controller

Consider a nonlinear plant associated with a fuzzy logic controller. Specifically, the linguistic sets of input and output of the fuzzy logic controller are positive big (PB), positive small (PS), zero (ZE), negative small (NS), and negative big (NB). The fuzzy rules are as follows: (i) If the input is PB, then the output is PB. (ii) If the input is PS, then the output is PS. (iii) If the input is ZE, then the output is ZE. (iv) If the input is NS, then the output is NS. (v) If the input is NB, then the output is NB. The output of the fuzzy logic controller is derived from the max-min compositional rule of inference and the center of gravity method based on membership functions of the linguistic sets mentioned above. Then the output of the fuzzy logic controller is symmetrical with respect to zero and bounded by a linear gain. Thus the nonlinearity of the fuzzy logic controller is a memoryless function that belongs to a given sector and the theorem becomes applicable to the fuzzy control system.

Example 1:

$$
\begin{aligned}
\dot{x_1} &= -\tfrac{1}{2}x_1 - x_2 + 2u \\
\dot{x_2} &= x_1 - 3x_2 - \tfrac{x_2^3}{4} \\
y &= \tfrac{x_1}{2} + x_2 \\
u &= -\phi(y)
\end{aligned}
\qquad (8)
$$

where the fuzzy logic controller, $\phi(\cdot)$, is described by the above fuzzy control algorithm with the membership functions shown in Fig. 2. The input-output relationship of the fuzzy logic controller is characterized by the sector [0,58] as shown in the right upper part of Fig. 2. As it follows from the theorem 1, the equilibrium at the origin is asymptotically stable for all $k \in [0, \infty)$ and its region of attraction is $\{x \in R^2 : \|x\|_2 < \infty\}$. Fig. 3 shows the phase portrait of (8). It is observed from Fig. 3 that the origin is asymptotically stable.

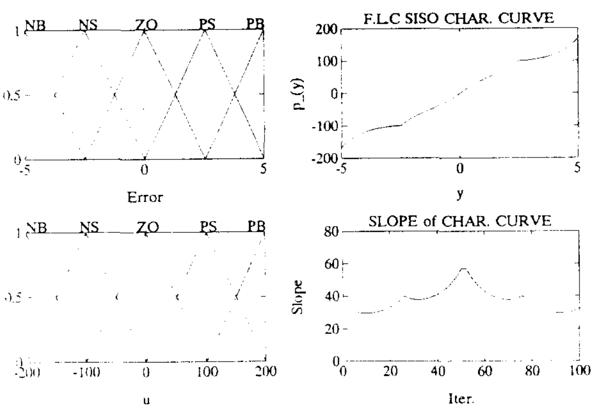

Fig. 2. Membership funtions, input-output relationships, and its slopes.

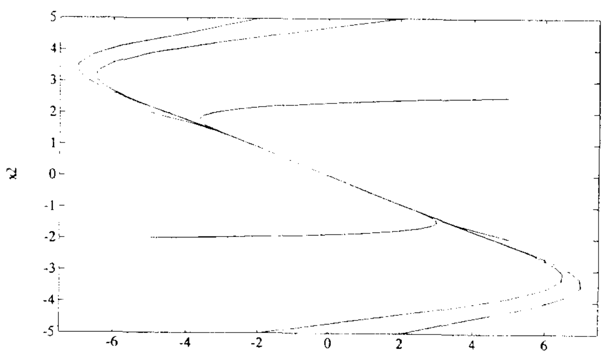

Fig. 3. Phase portrait of (8)

III. FINITE SECTOR BOUND CRITERION

1. Stability Criterions

If the conditions of the asymptotic globally stability in (1) – (3) are not met, we show the following theorem for asymptotic stability.

Theorem 2 Given the system defined by (1) – (3), assume that there exists a positive definite matrix P such that the matrix $R(x) = -(A^T(x)P + PA(x))$ is positive definite in some neighborhood Ω of the origin of R^n where $A(x) = \frac{\partial f(x)}{\partial x}$. Then the equilibrium point at the origin is asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[0, k]$ if there exists $\beta \geq 0$ such that for all $x \neq 0$ in Ω,

$$(\frac{1}{k} + \beta\frac{\partial h(x)}{\partial x}g(x))f^T(x)R(x)f(x) - \gamma^2 > 0$$

where $\gamma = (g^T(x)A^T(x)P - \frac{\beta}{2}\frac{\partial h(x)}{\partial x})f(x) - \frac{h(x)}{2}$. If $\Omega = R^n$ and $f^T(x)f(x)$ is radially unbounded, then the equilibrium point at the origin is globally asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[0, k]$.

Proof of Theorem 2: Consider

$$
\begin{aligned}
V &= f^T(x)Pf(x) + \beta\int_0^y \phi(\sigma)d\sigma \\
\text{Then } \dot{V} &= f^T(A^TP + PA)f + 2uf^TPAg + \beta\phi(y)\dot{y} \\
&= -f^TRf + 2uf^TPAg - \beta u\frac{\partial h}{\partial x}(f + gu) \\
\text{Since } & -uy - \frac{u^2}{k} + u(y + \frac{u}{k}) = 0, \\
\dot{V} &= -f^TRf + 2u(f^TPAg - \frac{h}{2}) - \frac{u^2}{k} \\
&\quad + u(y + \frac{u}{k}) - \beta u\frac{\partial h}{\partial x}(f + gu) \\
\text{Let } \gamma &= (g^TA^TP - \frac{\beta}{2}\frac{\partial h}{\partial x})f - \frac{h}{2}, \\
\text{then } \dot{V} &= -f^TRf + u(y + \frac{u}{k}) - [u\ \gamma]\Theta[u\ \gamma]^T + \frac{\gamma^2}{\zeta} \\
\text{where } \Theta &= \begin{bmatrix} \zeta & -1 \\ -1 & \frac{1}{\zeta} \end{bmatrix}, \quad \zeta = \frac{1}{k} + \beta\frac{\partial h}{\partial x}g
\end{aligned}
$$

Since R is positive definite and $\zeta f^TRf - \gamma^2 > 0$, ζ is positive and Θ is positive semi-definite. As it follows from that $u(y + \frac{u}{k}) \leq 0$ and $\zeta f^TRf - \gamma^2 > 0$, $\dot{V}$ is negative definite. Q.E.D.

For the decoupled system as (5) – (7), we have more simple criterion in the following corollary.

Corollary 2 Given the system defined by (5) – (7), assume that there exists a positive definte symmetric matrix P such that the matrix $Q(x) = -(F^T(x)P + PF(x))$ is positive definite in some neighborhood Ω of the origin of R^n. Then the equilibrium point at the origin is asymptotically stable for all $\phi(\cdot)$ belonging to the sector $[0, k]$ if there exists a $\beta \geq 0$ such that $\frac{1}{k} + \frac{\beta}{2}(\frac{\partial y}{\partial x})g(x) - v^T(x)Q^{-1}(x)v(x) > 0$, where $v(x) = Pg(x) - \frac{\beta}{2}F^T(x)(\frac{\partial y}{\partial x})^T$. If $\Omega = R^n$, then the equilibrium point at the origin is globally asymptotically stable for all $\phi(\cdot)$ belonging to the sector [0,k].

If we take $V = x^TPx + \beta\int_0^y \phi(\sigma)d\sigma$, the proof of the above is analogous to the theorem 2, so it is omitted here.

2. Stability Analysis with Fuzzy Logic Controller

If we take the fuzzy logic controller $\phi(y)$, as in the part 2 of section II, we can apply the above theorem 2. Consider the following system.

$$\text{Example 2:} \quad \begin{aligned} \dot{x}_1 &= -10x_1 - x_1^3 \\ \dot{x}_2 &= x_1^2 - 2x_2 + u \\ y &= 3.5x_1^2 \\ u &= -\phi(y) \end{aligned} \qquad (9)$$

where the fuzzy logic controller, $\phi(\cdot)$, is described by the above fuzzy control algorithm with the membership functions shown in Fig. 4. The input-output relationship of the fuzzy logic controller is characterized by the sector [0,30] as shown in the right upper part of Fig. 4. As it follows from the theorem 2, the equilibrium point at the origin is asymptotically stable for all $k \in [0, 50]$ and its region of attraction is $\{x \in R^2 : \|x\|_2 \leq 30\}$. Fig. 5 shows the phase portrait of (9). It is observed from Fig. 5 that the origin is asymptotically stable.

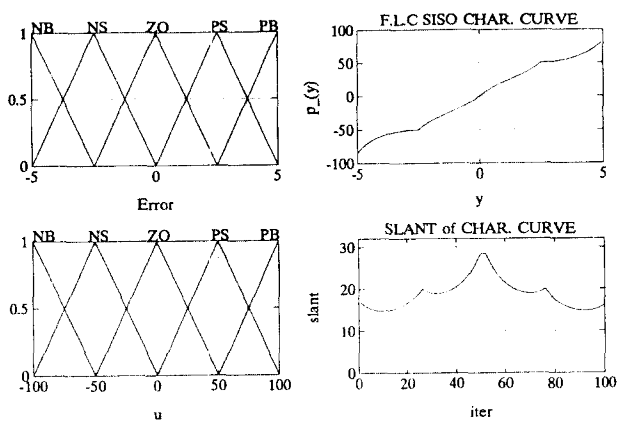

Fig. 4. Membership functions, input-output relationships, and its slopes.

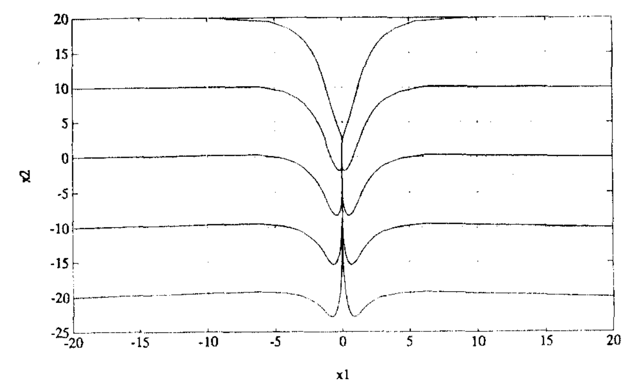

Fig. 5. Phase portrait of (9)

IV. CONCLUSIONS

In this paper, we have formulated algebraic conditions for the absolute stability of nonlinear plants with sector bounded nonlinear feedback. Since the nonlinearity of fuzzy logic controller can be characterized by sector bounded nonlinearity, the stability of fuzzy control systems is analyzed using the technique obtained in this paper. Conversely, when designing fuzzy logic controller, these criterions can also be used as a guideline for ensuring stability.

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