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STABILITY OF FUZZY DYNAMIC CONTROL SYSTEMS:

The Cell-State Transition Method

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Abstract

The objective of this paper is to provide fuzzy control designers with a **design tool** for **stable** fuzzy logic controllers. Given multiple sets of data disturbed by vagueness uncertainty, we generate the implicative rules that guarantee **stability** and **robustness** of closed-loop fuzzy dynamic systems. We propose the **cell-state transition method** which utilizes Hsu's **cell-to-cell mapping** concept [1]. As a result, a **generic** and **implementable** design methodology for obtaining a fuzzy feedback gain **K**, a fuzzy hypercube [2], is provided and illustrated with simple examples.

1 Introduction

Fuzzy logic control has been cast by many investigators as an expert system paradigm where human-like intelligence may be adopted and applied to some complex systems. This fuzzy logic artificial intelligence setting adjusts controller parameters or membership functions based on specified performance characteristics. We introduce a **systematic design** procedure for stabilizable fuzzy linguistic controllers. This method can be applied to a class of nonlinear systems which suffer from vagueness uncertainty.

One approach is to divide the uncertain objects or domains into a finite number of manageable quantities and to assign fuzzy sets to each linguistic representation; then, the relations that govern the control objectives may be obtained. Thus, we are dealing with quantitative objects of an infinite point space in terms of qualitative reasoning of finite rules. In this paper, the objects are the vector fields of the invariant manifolds or the switching manifolds in a hyperspace. We gather these data so that a controller based on the fuzzy set theoretic technique can be obtained.

Since fuzzy set theory includes binary set theory [3], it is more convenient to design and analyze fuzzy logic control systems on the basis of classical control theoretic techniques, and to map the appropriate crisp domains into corresponding fuzzy sets [4,5]. However, heuristics still take part in the design of fuzzy logic controllers, if *a priori* knowledge of the process is incomplete [6]. Especially, the size of a quantized subspace is another heuristic design parameter. Here, a **cell-group** is defined as a collection of quantized cells or elements which result from the α-cut of the corresponding fuzzy set. (for example, $\alpha = 0.5$). The cell-to-cell mapping concept in fuzzy logic has already been investigated and analyzed in [7]. We adopt and enhance this cell-state mapping technique so as to design the membership functions of a fuzzy controller for stabilizing a closed-loop fuzzy dynamic system. Our design technique has the following characteristics:

- Design of Scheduling in Crisp Domain ⇒ Fuzzification
- Control Rules as Cell-State Transitions
- Fuzzy Optimal Strategies (Min. Energy, Min. Squared Error, Min. Time, or Combined One)
- Use of Fuzzy Hypercubes (Multivariable/Multilayered FAM)

2 Fuzzy Dynamic Systems (FDS)

2.1 Dynamic Behavior of FDS

The primary objective of this section is to describe the dynamic behavior of discrete-time single-input single-output (SISO) fuzzy dynamic systems (FDS) and to show how to design a fuzzy logic controller with stability. A FDS can be represented in terms of a number of rules, such as

> **IF ('Input r_k' IS POSITIVE SMALL) AND ('State x_k' IS NEGATIVE LARGE) THEN 'State x_{k+1}' IS NEGATIVE LARGE.**
>
> ⋮ ⋮

where the subscript k denotes the time step. The block diagram of a fuzzy dynamic control system is shown in Figure 1. The rules are disjunctive with one another. x_{k+1} is inherently hidden inside the block of a FDS together with the one-step shift operator z^{-1} which is defined as $x_k = z^{-1}x_{k+1}$. A SISO FDS consists of a one-step time delay and a rulebase that deals with its dynamic state. In order to design a fuzzy logic controller, let us first consider a homogeneous fuzzy system without the input r_k.

A homogeneous FDS is expressed in the form [8]:

$$x_{k+1} = A \circ x_k \qquad x_0 \in \mathcal{X}. \tag{1}$$

where $\circ$ is the max-min operator and A is a square fuzzy relational matrix. In fact, u_k is a fuzzy variable in the rulebase of the above FDS, and x_k is defined as a fuzzy vector in $\mathcal{X}$, a family of fuzzy subsets. Now, a fuzzy subset L is defined as

$$L = \{(u_k, \mu_L(u_k)) | u_k \in U, L \in \mathcal{X}\}, \qquad x_k \triangleq \mu_L(u_k) \tag{2}$$

where U is the universe of discourse for the variable u_k. Each fuzzy subset L is associated with a linguistic element in T_x, a set of linguistic terms for x_k representing the vagueness uncertainty. Therefore, the function $\mu_L(\cdot)$ determines the membership of u_k, i.e., the shape and location of x_k in $\mathcal{X}$. For example, $x_k =$'$u_k$is Medium' may be denoted as follows:

$$x_k = [0\ \ 0.2\ \ 0.6\ \ 1\ \ 0.6\ \ 0.2\ \ 0]^T \tag{3}$$

Each element in x_k implies the degree of membership at the associated representative point in U. Fuzzification deals with quantization and assignment of a degree of membership for each quantized element. Again, asymptotic stability is addressed by considering

$$x_k = A \circ x_{k-1} = \cdots = A^k \circ x_0 \tag{4}$$

where A^k is defined as a series of max-min operation

$$A^k = \underbrace{A \circ A \circ \cdots \circ A}_{\leftarrow\, k\ A\text{'s}\, \rightarrow} \tag{5}$$

The homogeneous FDS, $x_{k+1} = A \circ x_k$, can be obtained by eliminating the input r_k in Figure 1. A linguistic representation of a single-premise single-consequence (SPSC) fuzzy relation can be described as

> **IF u_k IS POSITIVE LARGE, THEN v_k IS NEGATIVE SMALL.**

where $x_k^{PL} := \mu_{PL}(u_k)$; $y_k^{NS} := \mu_{NS}(v_k)$; and PL, NS stand for 'POSITIVE LARGE' and 'NEGATIVE SMALL', respectively. Note that $v_k = u_{k+1}$. This system becomes the FDS (1) when a delay is inserted between x_k and y_k so that $y_k = x_{k+1}$ holds. If $y_k = x_{k+1}$, linguistic connections within the rules can describe the dynamic behavior of the fuzzy system (1). For example, if there exists a positive integer n such that $A^{n+1} = A^n$ then we can say that x_k settles to a steady-state fuzzy vector x_∞.

The fuzzy relational matrix A is determined in terms of the rules describing the system's dynamic behavior. Let ⊗, ⊕ denote the minimum and maximum operators, respectively, then A is obtained according to

$$A = \oplus_{i=1}^{s} \{y^{Ly_i} \otimes x^{Lx_i T}\} \tag{6}$$

where y^{Ly_i} is the consequence fuzzy vector for the membership function of the i-th rule (for v_k), and $x^{Lx_i T}$ is the transpose of the associated premise fuzzy vector for the i-th rule (for u_k).

EXAMPLE 2.1: Let us consider two rules that govern the homogeneous FDS in (1):

RULE NO.1: IF u_k is PL THEN u_{k+1} is NL.
RULE NO.2: IF u_k is NL THEN u_{k+1} is PL.

If we choose $x_k^{PL} = [0\ \ 0.2\ \ 0.5\ \ 0.8\ \ 1.0]^T$ and $x_k^{NL} = [1.0\ \ 0.8\ \ 0.5\ \ 0.2\ \ 0]^T$ then the A matrix can be represented as

$$\begin{aligned} A &= (y^{NL_1} \otimes x^{PL_1 T}) \oplus (y^{PL_2} \otimes x^{NL_2 T}) \\ &= \begin{bmatrix} 0 & 0.2 & 0.5 & 0.8 & 1.0 \\ 0.2 & 0.2 & 0.5 & 0.8 & 0.8 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.8 & 0.8 & 0.5 & 0.2 & 0.2 \\ 1.0 & 0.8 & 0.5 & 0.2 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

Let the initial fuzzy vector be $x_0 = [0\ \ 0.3\ \ 0.5\ \ 0.7\ \ 1.0]^T$, then the subsequent x_k's are obtained using the max-min operations:

$$
\begin{aligned}
x_1 &= A \circ x_0 = [1.0\ 0.8\ 0.5\ 0.5\ 0.5]^T \\
x_2 &= A \circ x_1 = [0.5\ 0.5\ 0.5\ 0.8\ 1.0]^T \\
x_3 &= A \circ x_2 = [1.0\ 0.8\ 0.5\ 0.5\ 0.5]^T \\
x_4 &= \cdots
\end{aligned}
$$

In this example, x_k oscillates with a period of $T = 2$ and note that $A^{k+2} = A^k$. Therefore, this FDS can emulate periodic responses as in crystal oscillators. □

In Example 2.1, it is shown that this simple FDS can generate oscillating waveforms. This fuzzy relation can be sought in the structure of the linguistic rules and their membership functions. Linguistic memberships may be regarded as nodes, and the relational matrix, in a fuzzy sense, results from transitions from one such node to another. In brief, these **linguistic membership transitions** govern the behavior of a FDS in an oscillatory, underdamped, overdamped, or even unstable mode. Every rule in the control knowledge base is a state transition from one membership (cell-group) to another.

EXAMPLE 2.2: Let a fuzzy logic controller have three rules, R(i) through R(iii):

$$
\begin{aligned}
&\text{R(i):} && (x_k^{PL}) \Longrightarrow (y_k^{ZE}). \\
&\text{R(ii):} && (x_k^{NL}) \Longrightarrow (y_k^{ZE}). \\
&\text{R(iii):} && (x_k^{ZE}) \Longrightarrow (y_k^{ZE}). \qquad (*)
\end{aligned}
$$

where $\Longrightarrow$ denotes the 'IF-THEN' implication, x_k^{PL} stands for 'u_k IS POSITIVE LARGE', y_k^{ZE} abbreviates the expression 'v_k IS ZERO', and x_k^{NL} represents 'u_k IS NEGATIVE LARGE'. Let the universe of discourse have nine elements and let the membership functions (fuzzy vectors) be denoted as follows:

$$
\begin{aligned}
x_k^{PL} &= [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0.5\ 1]^T \\
x_k^{ZE} &= [0\ 0\ 0\ 0.5\ 1\ 0.5\ 0\ 0\ 0]^T \\
x_k^{NL} &= [1\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T.
\end{aligned}
$$

The graphical representation is shown in Figure 2. Note that the transition (*) is a stable equilibrium condition for invariant manifolds of differential dynamic systems. Let us consider the output of the homogeneous FDS.

CASE) Let the initial condition x_0 be

$$x_0 = [0.5\ 1\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0]^T$$

then the subsequent x_k's are obtained by applying the max-min operation

$$
\begin{aligned}
x_1 &= A \circ x_0 = [0\ 0\ 0\ 0.5\ 0.5\ 0.5\ 0\ 0\ 0]^T \\
x_2 &= A \circ x_1 = x_1
\end{aligned}
$$

and it is easy to recognize that $u_k = 0$ for $k \geq 1$ as the steady-state value of defuzzification. □

2.2 Cell-State Transition Method via α-Cut Cell-Groups

As we discussed in the previous section, the membership transitions are equivalent to the cell-state transitions if we consider 'mutually exclusive' cell-groups via α-cuts or other in-between cuts. Hsu's cell-to-cell mapping concept [1] is utilized here to show how arbitrary initial states are forced to move towards the desired goal state. Preliminary analysis of the cell-state mapping theory has been given in [7]. We investigate more detailed relationship between transitions in the cell-state space and rules in a fuzzy hypercube [2]. For convenience, we will simplify the quantization and fuzziness of the membership functions by utilizing α-cut cell-groups. In our case, the membership values of x_k^{Lx} and y_k^{Ly} are to be reduced to bold **1**'s in the matrix A, if the grades of both membership functions exceed the same α (for example $\alpha = 0.5$). When choosing α, a larger α implies that emphasis is placed on mutual exclusiveness. On the other hand, we take the membership values as bold zeros, **0**, if the grades are less than the chosen α. For example, a fuzzy vector given as

$$x^L = [0\ 0\ 0\ 0\ 0.2\ 0.4\ 0.6\ 0.8\ 1]^T$$

can be written, with $\alpha = 0.5$, as

$$x_\alpha^{PL} = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1]^T$$

which can be regarded as making a crisp set out of a fuzzy set, as an α-cut [9]. The above characteristic function x_α^{PL} may be further reduced to

$$x_\alpha^{PL} = [\mathbf{0}\ \ \mathbf{0}\ \ \mathbf{1}]^T$$

by grouping two 'three zeros' and one 'three ones'. Bold ones or zeros represent cell-groups.

Consider the following seven rules that represent an underdamped case for FDS's:

$$
\begin{aligned}
&\text{R(1):} && (x_k^{L_3}) \Longrightarrow (y_k^{L_{-2}}). \\
&\text{R(2):} && (x_k^{L_{-3}}) \Longrightarrow (y_k^{L_2}). \\
&\text{R(3):} && (x_k^{L_{-2}}) \Longrightarrow (y_k^{L_1}). \\
&\text{R(4):} && (x_k^{L_2}) \Longrightarrow (y_k^{L_{-1}}). \\
&\text{R(5):} && (x_k^{L_{-1}}) \Longrightarrow (y_k^{L_0}). \\
&\text{R(6):} && (x_k^{L_1}) \Longrightarrow (y_k^{L_0}). \\
&\text{R(7):} && (x_k^{L_0}) \Longrightarrow (y_k^{L_0}). \qquad (*)
\end{aligned}
$$

Each rule constitutes a cell-state transition of a cell-to-cell mapping defined by $\mathbf{F(Z)}$ in [7]. As shown in Figure 3, the dynamic response of a cell $z_1 \in Z$ moves to another cell $z_2 \in Z$ under the map $\mathbf{F}$ within time Δt. Again, rule R(7) is a rule of state invariance (*), and the dynamics of this fuzzy system reveal oscillatory responses with an asymptotic behavior. The membership transitions using the α-cuts are shown in Figure 4, and the matrix A is obtained as

$$
A = \oplus_{i=1}^{7} A_i = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{(1)} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{(4)} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{1}_{(5)} & \mathbf{1}_{(7)} & \mathbf{1}_{(6)} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{1}_{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{1}_{(2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} \qquad (8)
$$

where the subscripts in the boldface notation **1** denote the rule number. Let us treat these boldface **1**'s and **0**'s as regular numbers for the sake of simplicity.

CASE) Let the initial fuzzy vector x_0 be

$$x_0 = [0\ 0\ 0\ 0\ 0\ 0.5\ 1]^T$$

where 0.5 denotes a cell-group consisting of those cells whose grade of membership is less than α and greater than β ($\beta < \alpha$). Then, the subsequent x_k's are

$$
\begin{aligned}
x_1 &= A \circ x_0 = [0\ 1\ 0.5\ 0\ 0\ 0\ 0]^T \\
x_2 &= A \circ x_1 = [0\ 0\ 0\ 0.5\ 1\ 0\ 0]^T \\
x_3 &= A \circ x_2 = [0\ 0\ 0\ 1\ 0\ 0\ 0]^T \\
x_4 &= A \circ x_3 = x_3.
\end{aligned}
$$

Therefore, the FDS has reached a steady-state vector, and a steady-state value $u_k = 0$ after defuzzification. Note that $x_4 = A^4 \circ x_0$. The defuzzified values of x_k exhibit an underdamped response. it is obvious that $A^4 = A^3$ and the matrix A^3 has 1's in the 4-th row and 0's elsewhere. This matrix plays an important role in the stability analysis of membership transitions in FDS's. In general, there exists a positive integer k such that $A^{k+1} = A^k$ and the k-th max-min power of the A matrix becomes the same α-cut pattern (1's in the middle row) for every FDS with asymptotic stability. Now we consider the following theorems in order to describe a class of FDS's in general:

THEOREM 2.1: A homogeneous FDS is **asymptotically stable** with respect to an equilibrium membership vector x_e if and only if there exists an invariant membership transition (invariance rule), 'IF x_e THEN x_e' (or '$x_e \Longrightarrow x_e$' in shorthand), and there exist positive integers k, k_∞ such that

$$A^k \circ x_0 = x_e \qquad k \geq k_\infty$$

holds for all initial membership vectors x_0. We define $A^k = A \circ \cdots \circ A$ as the fuzzy transition operator. □

THEOREM 2.2: A homogeneous FDS is **periodic** with period T (T is a positive integer) if and only if there exists a fuzzy transition operator A^T such that

$$A^T \circ x = x \qquad \& \qquad A^k = A^{k+T}$$

are satisfied for all $k > 0$ and for some membership vector x. □

EXAMPLE 2.3: An oscillatory FDS may be described using four rules of membership transition. In this example, there are four quantized cell-groups (linguistically L_{-2}, L_{-1}, L_1, L_2) under consideration.

$$
\begin{aligned}
&\text{R(1):} && (x_k^{L_{-2}}) \Longrightarrow (y_k^{L_1}). \\
&\text{R(2):} && (x_k^{L_1}) \Longrightarrow (y_k^{L_2}). \\
&\text{R(3):} && (x_k^{L_2}) \Longrightarrow (y_k^{L_{-1}}). \\
&\text{R(4):} && (x_k^{L_{-1}}) \Longrightarrow (y_k^{L_{-2}}).
\end{aligned}
$$

The α-cut patterns in A are obtained as

$$
A = \oplus_{i=1}^{4} A_i = \begin{bmatrix}
0 & 1_{(4)} & 0 & 0 \\
0 & 0 & 0 & 1_{(3)} \\
1_{(1)} & 0 & 0 & 0 \\
0 & 0 & 1_{(2)} & 0
\end{bmatrix}.
$$

For any initial membership vectors, A has the following property for a period of $T = 4$: $A^4 \circ x = x$, $\forall x \in \mathcal{X}$ and $A^{k+4} = A^k$, $\forall k \geq 0$. The graphical representation of the cell-state transitions in Example 2.3 is shown in Figure 5.

EXAMPLE 2.4: Now let us examine another FDS with asymptotic stability. The state is decomposed into seven quantised cell-groups. Cell-state transitions between the seven cell-groups constitute the following seven rules:

R(1): $(x_k^{L_{-3}}) \Longrightarrow (y_k^{L_{-2}})$.
R(2): $(x_k^{L_{-2}}) \Longrightarrow (y_k^{L_{-1}})$.
R(3): $(x_k^{L_{-1}}) \Longrightarrow (y_k^{L_0})$.
R(4): $(x_k^{L_3}) \Longrightarrow (y_k^{L_2})$.
R(5): $(x_k^{L_2}) \Longrightarrow (y_k^{L_1})$.
R(6): $(x_k^{L_1}) \Longrightarrow (y_k^{L_0})$.
R(7): $(x_k^{L_0}) \Longrightarrow (y_k^{L_0})$. (*)

We assume that $y_k = x_{k+1}$. Note that, without rule $R(7)$, the particular cell-group representing L_0 cannot guarantee asymptotic stability, and, therefore, no dynamic description is possible for L_0. The above rules are shown as cell-state transitions in Figure 6.

Using α-cut patterns, A can be represented as

$$A = \oplus_{i=1}^{7} A_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1_{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{(2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{(3)} & 1_{(7)} & 1_{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{(5)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and similarly, after four max-min operations, we obtain

$$A^4 = A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (9)$$

and $x_4 = x_3 = [0\ 0\ 0\ 1\ 0\ 0\ 0]^T$ resulting in $u_3 = 0$. □

Of course, in Example 2.4, the behavior of the over-damped FDS can be made faster if the elements of A directly show the same pattern as in (9). However, the α-cut method is an extremely simplified method to demonstrate how a particular fuzzy dynamic system changes its state as the time step increases. It is also viewed as a mutually exclusive case of complexity in a fuzzy system according to Zadeh's explanation of 'complexity' as 'the principle of incompatibility' [10]. What interests us is that vagueness uncertainty in FDS's can be handled via overlapping mutually exclusive cell-groups in an excellent and smooth manner.

2.3 Stability of Fuzzy Dynamic Systems: SISO

In this section, we continue our design procedure by augmenting a control input γ_k to a homogeneous FDS described in the previous section. Again, the α-cuts are used for the sake of simplicity. The block diagram of a SISO fuzzy feedback control system is shown in Figure 1 where the feedback matrix K is introduced so that the closed-loop feedback system can be made stable. The objective of this section is to derive a fuzzy relation K that stabilizes our SISO FDS in Figure 1. The procedure for synthesizing K is summarized as follows:

- First, choose representative constant values γ for γ_k that exhibit every dynamic behavior of A. The membership vector for γ is r and is obtained from appropriate quantization. Assign the corresponding subscript γ for $A = A_\gamma$.
- Second, slices of the matrices A_γ constitute a fuzzy hypercube of dimension 3, $\mathcal{P}$, and the feedback matrix K includes rules of the following form:

  K_1: **IF** $x_k^{L_1}$, **THEN** $r_k^{U_1}$
  K_2: **IF** $x_k^{L_2}$, **THEN** $r_k^{U_2}$
  ⋮

  where L_i and U_i are the i-th linguistic terms for x_k and r_k, respectively. The desired matrix A^* is chosen so that asymptotic stability is satisfied using the fuzzy relational matrices A_γ's. Hence, K is obtained from

$$K = \oplus_i \{ r_k^{U_i} \otimes x_k^{L_i T} \}. \quad (10)$$

The desired matrix A^* is found via selection of an invariance rule and membership transition with the property of asymptotic stability, as discussed in the previous section. The following example shows how to design a fuzzy feedback K based on the above considerations.

EXAMPLE 2.5: Let a single-link telerobot arm be attached to the space shuttle as shown in Figure 7. The angle of the link with respect to the vertical axis is defined as the membership vector x_k. The zero angle is, in a fuzzy vector form, $x_k = [0\ 0\ 1\ 0\ 0]^T$. In the universe of discourse, the representative values for each interval of u_k are $[-30°, -15°, 0°, 15°, 30°]$. The dynamic behavior of the telerobot arm is described with 3 available control torque values $\gamma = -1, 0, 1$ as follows:

- $\gamma = -1$: $x_{k+1} = A_{-1} \circ x_k$ (cf. $x_{k+1} = f(x_k, -1)$).

$$A_{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1^*_{(1)} & 0 \\ 0 & 0 & 0 & 0 & 1^*_{(2)} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad r^{U_{-1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- $\gamma = 0$: $x_{k+1} = A_0 \circ x_k$ (cf. $x_{k+1} = f(x_k, 0)$).

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1^*_{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad r^{U_0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- $\gamma = 1$: $x_{k+1} = A_1 \circ x_k$ (cf. $x_{k+1} = f(x_k, 1)$).

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1^*_{(4)} & 0 & 0 & 0 & 0 \\ 0 & 1^*_{(5)} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad r^{U_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The desired matrix A^* consists of 1^*'s among the matrices A_{-1}, A_0, A_1:

$$A^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1^*_{(4)} & 0 & 0 & 0 & 0 \\ 0 & 1^*_{(5)} & 1^*_{(3)} & 1^*_{(1)} & 0 \\ 0 & 0 & 0 & 0 & 1^*_{(2)} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (11)$$

and therefore A^* forms a homogeneous FDS with asymptotic stability and overdamped characteristics. The linguistic terms for x_k are $\{L_{-2}, L_{-1}, L_0, L_1, L_2\}$.

The fuzzy feedback relational matrix K is obtained from the subscript (i) in $1^*_{(i)}$ that is also the index of the following rules:

K_1: $x_k^{L_1} \Longrightarrow r_k^{U_{-1}}$.
K_2: $x_k^{L_2} \Longrightarrow r_k^{U_{-1}}$.
K_3: $x_k^{L_0} \Longrightarrow r_k^{U_0}$.
K_4: $x_k^{L_{-2}} \Longrightarrow r_k^{U_1}$.
K_5: $x_k^{L_{-1}} \Longrightarrow r_k^{U_1}$.

Therefore, K is computed as

$$K = \{ r^{U_{-1}} \otimes x^{L_1 T} \} \oplus \cdots \{ r^{U_1} \otimes x^{L_{-1} T} \} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

and the membership vector of the control torque r_k is obtained from the max-min operation

$$r_k = K \circ x_k \quad (13)$$

and finally a crisp value is derived from

$$\gamma_k = \text{DEFUZZIFIER}(r_k) \quad (14)$$

where DEFUZZIFIER($\cdot$) is chosen from such defuzzification strategies as the maximum criterion, the mean of maxima procedure, and the centroid algorithm.

Let the initial condition for x_k be $x_0 = [1\ 0\ 0\ 0\ 0]^T$, then the control r_0 is obtained via the max-min operation

$$r_0 = K \circ x_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the next membership vector x_1 is

$$x_1 = \mathcal{P} \circ (x_0 \otimes r_0) = [0\ 1\ 0\ 0\ 0]^T.$$

In the next step, r_1 is computed as

$$r_1 = K \circ x_1 = [0\ 0\ 1]^T$$

and x_2 is calculated from

$$x_2 = \mathcal{P} \circ (x_1 \otimes r_1) = [0\ 0\ 1\ 0\ 0]^T.$$

After subsequent calculations, we conclude that $x_3 = x_2$ and the angle of the telerobot arm converges to zero asymptotically. □

3 Conclusions

We address the **cell-state transition method** which is a generalized approach to the design of fuzzy dynamic controllers. Stability is automatically treated in the design procedure of membership functions in the rulebase. With the examples, it is demonstrated that a fuzzy hypercube can be applied to stabilize the closed-loop fuzzy dynamic control systems via the cell-state transition method. Each rule in a hypercubic controller **K** constitutes a cell-state transition which forces any state trajectories to follow the predetermined path toward the goal state.

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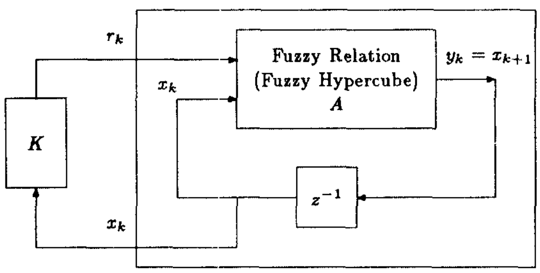

Figure 1: Fuzzy Dynamic Control System

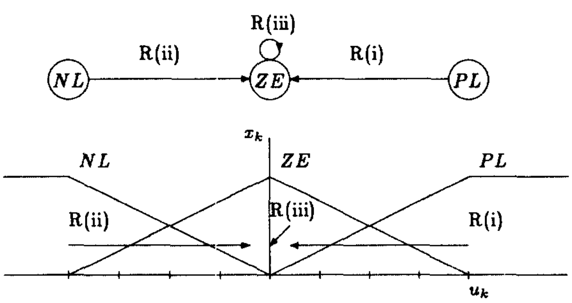

Figure 2: Membership Transitions of $x_{k+1} = A \circ x_k$

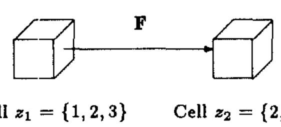

Figure 3: A Cell-State Transition from z_1 to z_2 in $\mathbf{Z}$

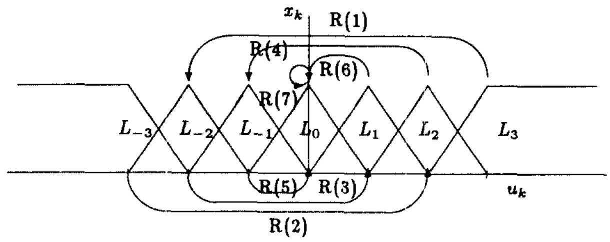

Figure 4: 7 Membership Transitions of $x_{k+1} = A \circ x_k$

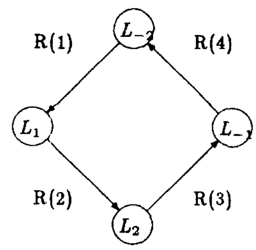

Figure 5: Cell-State Transitions of Period-4 Motion

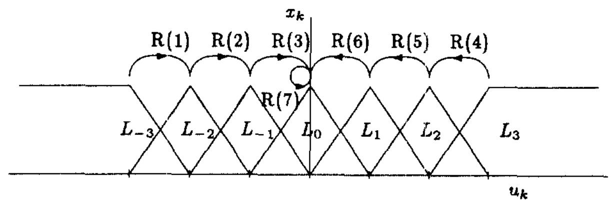

Figure 6: Seven Membership Transitions for Example 3.4

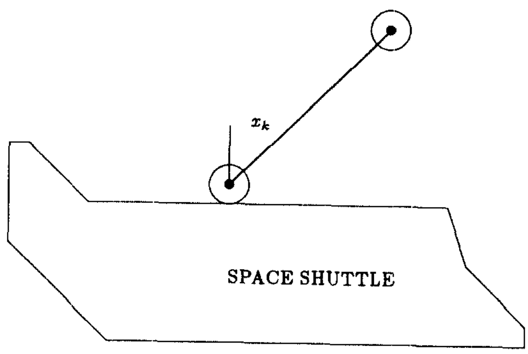

Figure 7: A Single-Link Telerobot Arm on A Space Shuttle