한국소음진동공학회 '93추계학술대회논문집 pp. 29 ~ 34.

A Strategy for Moving Mass Systems from One Point to Another without Inducing Residual Vibration

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Abstract

In many circumstances, it is desired to move a mass from one position to another without inducing any vibration in the mass being moved. Two such problems are considered here: the motion of a pendulum initiated by the specified motion of its support. In each case, it is desired that the system start at rest and come to rest in the second position. A simple strategy for the specified motion is given here. The method is motivated by engine cam-follower design. The force required to move the system in question is determined as well as the maximum value of the force required (and the times at which these forces take place) is determined.

Introduction

In many circumstances, it is desired to move a mass from one position to another without inducing any vibration in the mass being moved. Examples include valve train systems in engines, recording heads on computer disk drives and robot arms [1]. In fact, the general design of a typical engine cam gives the motivation for the specified motion we attempt below. There are several approaches to the problem posed here, see, for example [2,3].

Case Study for Dwell-Rise-Dwell Motion

Suppose we wish to move the mass m in Figure 1 by moving the cart in a prescribed motion. Suppose both the mass and cart are at rest initially. And suppose that we wish to move

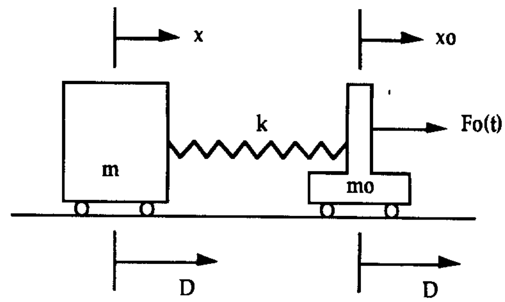

Figure 1 The System: Movement of a Mass

the cart so that the mass m comes to rest again after a displacement D.

The differential equation of motion of the mass m is:

$$F_x = ma_x : \quad k[x_o(t) - x] = m\ddot{x} \qquad (1)$$

or

$$\ddot{x} + \omega_n^2 x = \omega_n^2 x_o(t), \qquad (2)$$

where $\omega_n = \sqrt{k/m}$

Since we want m to move a distance D with no overshoot, it is important that the cart move a distance D as well. This means the spring which is unstressed initially will be unstressed in the final position as well. Thus each of the two masses will end up with a displacement D.

In order to accomplish this, let us suppose that the specified displacement has the form:

$$x_o(t) = \frac{1}{2}D[1 - \cos(\frac{\omega_n}{\alpha}t)] \qquad 0 \le t \le (\alpha\pi/\omega_n)$$
$$= D \qquad t > (\alpha\pi/\omega_n) \qquad (3)$$

The motivation for the (1 - cos θ) in (3) comes from typical shapes used in engine valve train systems [4]. A plot of (3) is shown in Figure 2 . Notice that the shape of the curve gives hope that the strategy might work. Notice also that the

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speed with which we can make the move is determined by α since π/ω_n is fixed.

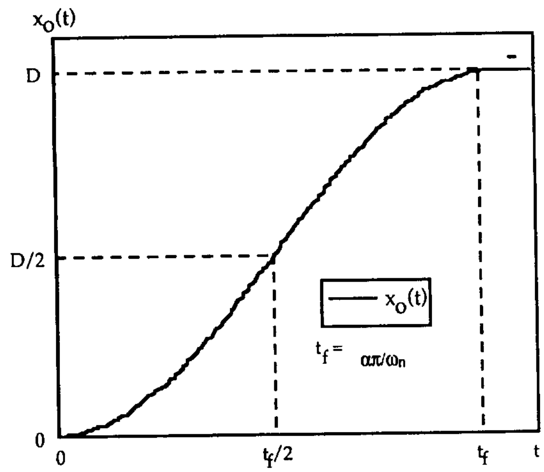

Figure 2 Prescribed Motion $x_o(t)$

We will see that the values of α which will work are:

$$\alpha = 3, 5, 7, \ldots . \tag{4}$$

That is, the frequencies of the prescribed motion $x_o(t)$ can be:

$$\omega_n/3, \omega_n/5, \omega_n/7, \ldots, \omega_n/(2n+1) \qquad n = 1, 2, 3, \ldots \tag{5}$$

We construct the solution to (2) with the specified displacement $x_o(t)$ given by (3):

$$\ddot{x} + \omega_n^2 x = \frac{1}{2} D\omega_n^2 [1 - \cos(\frac{\omega_n}{\alpha} t)] \tag{6}$$

The right hand side of (6) consists of a constant term plus a cosine term. As such, we guess a particular (forced) solution of the form:

$$x_p = P + Q\cos(\frac{\omega_n}{\alpha} t) \tag{7}$$

where P and Q are constants to be determined so that (6) is satisfied. Inserting (7) into (6) and canceling the common term (ω_n^2) gives:

$$P = \frac{1}{2} D \qquad Q = -\frac{1}{2} D(\frac{\alpha^2}{\alpha^2 - 1}) \tag{8}$$

The homogeneous solution to (3.2.6) has the form:

$$x_h = A\sin(\omega_n t) + B\cos(\omega_n t) . \tag{9}$$

Adding (7) and (9), we get:

$$x = \frac{1}{2} D[1 - \frac{\alpha^2}{\alpha^2 - 1} \cos(\frac{\omega_n}{\alpha} t)] + A\sin(\omega_n t) + B\cos(\omega_n t) \tag{10}$$

We determine the constants A and B from the initial conditions:

$$x(0) = \dot{x}(0) = 0 . \tag{11}$$

These conditions yield

$$A = 0, \qquad B = \frac{1}{2} D[\frac{1}{(\alpha^2 - 1)}] \tag{12}$$

Thus, we have the complete solution:

$$x = \frac{1}{2} D[1 - \frac{\alpha^2}{\alpha^2 - 1} \cos(\frac{\omega_n}{\alpha} t) + \frac{1}{\alpha^2 - 1} \cos(\omega_n t)] \tag{13}$$

There are two conditions which we must now impose on (13). First, we want $x(t_f) = D$ where $t_f = (\alpha\pi/\omega_n)$ And finally, we want the velocity of m to vanish at t_f (i.e. $\dot{x}(t_f) = 0$). Note that:

$$\frac{\omega_n}{\alpha} t_f = \pi \text{ rad. and } \omega_n t_f = \alpha\pi \text{ rad.} \tag{14}$$

Thus from (13), we get:

$$x(t_f) = \frac{1}{2} D[1 + \frac{\alpha^2}{\alpha^2 - 1} + \frac{1}{\alpha^2 - 1} \cos(\alpha\pi)] \tag{15}$$

The terms inside the [ ] must add to 2 if $x(t_f) = D$. This will happen if $\cos(\alpha\pi) = -1$. Thus, this condition requires:

$$\alpha = 3, 5, 7, \ldots, (2n+1) . \tag{16}$$

(Note that the term $\alpha = 1$ is inappropriate.)

Now we require that $\dot{x}(t_f) = 0$. From (13) and (14), we get

$$\dot{x}(t_f) = -\frac{1}{2} \omega_n D[\frac{1}{\alpha^2 - 1} \sin(\alpha\pi)] \tag{17}$$

This expression vanishes if $\alpha = 2, 3, 4, \ldots$.

The terms which are common to (16) and (17) are then the odd integer values of α from 3 on. Thus the solution for the motion of the mass m is:

$$\begin{aligned} x &= \frac{1}{2} D[1 - \frac{\alpha^2}{\alpha^2 - 1} \cos(\frac{\omega_n}{\alpha} t) + \frac{1}{\alpha^2 - 1} \cos(\omega_n t)], \quad 0 \le t \le t_f \\ &= D \qquad t > t_f \end{aligned} \tag{18}$$

where $t_f = (\frac{\alpha\pi}{\omega_n})$ and $\alpha = 3, 5, 7, \ldots, (2n+1)$.

A plot of $x_o(t)$ and $x(t)$, the specified motion of the cart and the resulting motion of m, is shown in Figure 4 . Notice that at t_f the distance between m and the cart is the same as it was initially. Thus the spring is unstressed. And since the velocity of m is zero and the prescribed motion of the cart remains at $x_o = D$, the mass m will remain at rest at the point $x = D$.

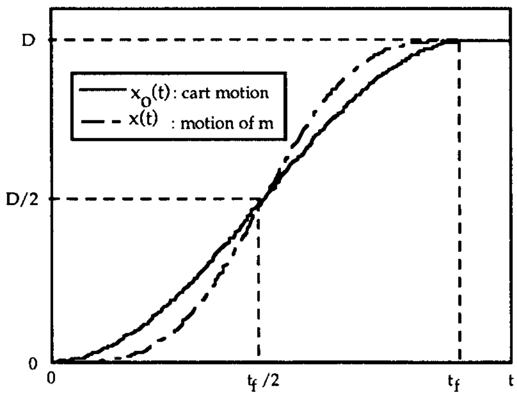

Figure 3 The Motion $x_o(t)$ and the Mass Motion x(t)

Equation (18) gives the design equation for the motion shown in Figure 3. As noted, α can take the values 3, 5, 7, ... Thus the time required to move the mass m a distance D is:

$$t_f = (\frac{\alpha\pi}{\omega_n}) \quad (19)$$

In order to minimize the time t_f, we should select $\alpha = 3$ and select a large value of the spring constant k (to increase the natural frequency $\omega_n = \sqrt{k/m}$). The solution is not unique since we can take $\alpha = 3, 5, 7$ ... and adjust ω_n accordingly.

The design equations for the strategy are contained in equations (3) and (18). One final consideration is the force $F_o(t)$ required to move the cart. Writing $F_o = m_o a_o + k(x_o - x)$ where a_o is the second (time) derivative of the motion (3), we obtain:

$$\frac{2F_o}{Dk} = [(\frac{m_o}{m\alpha^2}) + \frac{1}{(\alpha^2-1)}]\cos(\frac{\omega_n}{\alpha}t) - \frac{1}{\alpha^2-1}\cos(\omega_n t)]$$

where $0 \le t \le \frac{\alpha\pi}{\omega_n}$ (20)

The maximum amplitude of $\frac{2F_o}{Dk}$ depends on α and the mass ratio $\frac{m_o}{m}$. Suppose we take $\alpha = 3$ (the case for the fastest transit from A to B). Then we can determine the positions and heights of the maximum amplitudes:

The initial and final amplitude of $\frac{2F_o}{Dk}$ is also a function of the mass ratio $\frac{m_o}{m}$:

$$\left(\frac{2F_o}{Dk}\right)\Bigg|_{t=0} = (\frac{m_o}{m\alpha^2}) = (\frac{R_m}{\alpha^2}) \quad (21)$$

A plot of $\frac{2F_o}{Dk}$ is shown in Figure 4. Here $\alpha = 3$ and $\frac{m_o}{m} = 0.20$.

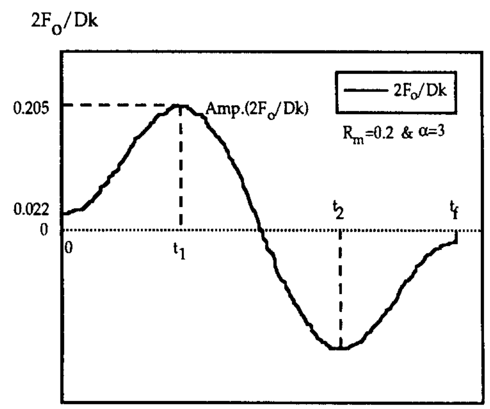

Figure 4 The Force Required to Initiate the Motion $x_o(t)$ for the Movement of a Mass

Case Study for the Pendulum Motion

Suppose now that we have a simple pendulum whose support o is to be moved a distance D. Suppose at t = 0, the pendulum is at rest : $\theta = \dot{\theta} = 0$. We then seek a strategy for moving the support o in a prescribed fashion so that when o has moved the distance D, the pendulum again comes to rest. See Figure 5.

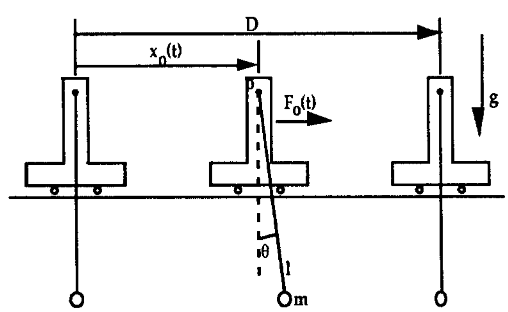

Figure 5 The System: Movement of a Simple Pendulum

The situation in Figure 5 could be a model for a crane system designed to move material from one point to another. The amount of material moved is unimportant since the design parameter $\omega_n = \sqrt{\frac{g}{l}}$ does not depend upon the mass of the pendulum.

We denote the prescribed motion of o by $x_o(t)$. Writing **F** = ma in the direction perpendicular to the pendulum string, we have:

$$-mg \sin\theta = ml\ddot{\theta} + m\ddot{x}_o \cos\theta \qquad (22)$$

Suppose the angle θ remains small so that we can make the approximations:

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1$$

Thus we have a differential equation for the pendulum:

$$\ddot{\theta} + \omega_n^2 \theta = -\frac{\ddot{x}_o}{l} \qquad (23)$$

where $\omega_n = \sqrt{\frac{g}{l}}$

Suppose that we take the same displacement function that we chose for the problem in Figure 2:

$$x_o(t) = \frac{1}{2}D[1-\cos(\frac{\omega_n}{\alpha})t] \qquad 0 \le t \le (\alpha\pi/\omega_n)$$
$$= D \qquad t > (\alpha\pi/\omega_n) \qquad \text{bis } (3)$$

From (23), we need the second (time) derivative of $x_o(t)$. Thus (23) becomes:

$$\ddot{\theta} + \omega_n^2 \theta = -\left(\frac{D\omega_n^2}{2\alpha^2 l}\right)\cos\left(\frac{\omega_n}{\alpha}t\right)$$
$$0 \le t \le (\alpha\pi/\omega_n) \qquad (24)$$

The initial conditions are:

$$\theta = \dot{\theta} = 0 \qquad (25)$$

For a particular solution to (24), we try:

$$\theta_p = Q \cos\left(\frac{\omega_n}{\alpha}t\right) \qquad (26)$$

Inserting this in (24) and canceling the common cosine terms, we get:

$$\left(1-\frac{1}{\alpha^2}\right)Q\omega_n^2 = -\frac{D\omega_n^2}{2\alpha^2 l}$$

or

$$Q = -\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right) \qquad (27)$$

Adding the particular solution to the homogeneous solution, we get:

$$\theta(t) = -\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right)\cos\left(\frac{\omega_n}{\alpha}t\right)$$
$$+A \sin(\omega_n t) + B \cos(\omega_n t) \qquad (28)$$

Imposing the conditions (25), we determine:

$$A = 0, \qquad B = \frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right) \qquad (29)$$

Finally, we have the motion $\theta(t)$:

$$\theta(t) = \frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right)\left[\cos(\omega_n t) - \cos\left(\frac{\omega_n}{\alpha}t\right)\right] \qquad 0 \le t \le t_f$$
$$= 0 \qquad t > t_f \qquad (30)$$

At this point, we must determine the values of α which give

$$\theta(t_f) = \dot{\theta}(t_f) = 0 \qquad (31)$$

where $t_f = \left(\frac{\alpha\pi}{\omega_n}\right)$

Note that $\omega_n t_f = \alpha\pi$ and $\frac{\omega_n t_f}{\alpha} = \pi$. Thus from (30), we obtain

$$\theta(t_f) = \frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right)[\cos(\alpha\pi)+1]$$

In order that $\theta(t_f) = 0$, we set $\cos(\alpha\pi) = -1$. Thus:

$$\alpha = 1, 3, 5, 7 \ldots$$

Differentiating (24) and evaluating at t_f , we obtain:

$$\dot{\theta}(t_f) = \frac{D}{2l}\left(\frac{\omega_n}{\alpha^2-1}\right)\left[-\sin(\omega_n t_f) + \left(\frac{1}{\alpha}\right)\sin\left(\frac{\omega_n}{\alpha}t_f\right)\right]$$

Noting again that $\omega_n t_f = \alpha\pi$ and $\frac{\omega_n t_f}{\alpha} = \pi$, if α = 1, 2, 3, ... the sine terms inside the brackets vanish. Thus the values of θ and $\dot{\theta}$ are zero at $t = t_f$ as desired. Once again, we note that α = 1 makes the denominator in (30) vanish. Thus acceptable values of α are 3, 5, 7, ... , (2n+1)

The prescribed motion $x_o(t) = 0$ and the response $\theta(t)$ are shown in Figure 6. In order to fully understand the results shown in Figure 6, we must compute the maximum amplitude of the response, θ_{max}.

If α = 3, it can be determined that $|\theta_{max}|$ occurs at t = 0.304 t_f and 0.696 t_f. Plugging either value in (30), we obtain:

$$\theta_{max} = 0.096\frac{D}{l} \qquad (32)$$

and again, the final time is:

$$t_f = \frac{3\pi}{\omega_n} \qquad (\alpha = 3) \qquad (33)$$

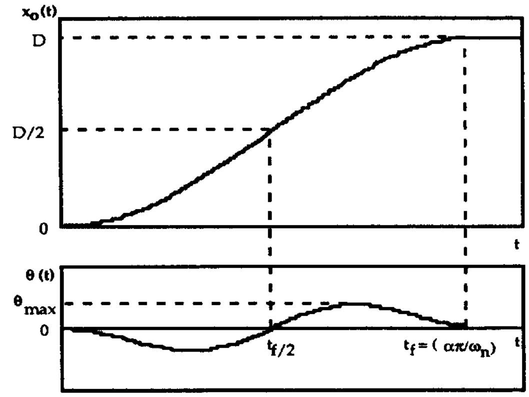

Figure 6 The Motion $x_o(t)$ and the Pendulum Motion θ (t)

It is important to note that while α = 3 gives the minimum time to move the system a distance D as well as the smoothest motion, the price which is paid is that the maximum angle θ occurs at this value of α. If we select α = 5, 7, ... , (2n+1) the maximum angle θ_{max} will be reduced, but the time t_f will be extended and the motion will involve higher harmonics not seen in the case α = 3.

Suppose that we generalize the discussion by considering a compound pendulum instead of a simple pendulum. See Figure 7. We will find that the fundamental design equations are essentially the same as above. In addition, we consider the force $F_o(t)$ required to generate the motion.

From Figure 7 we can determine that the two equations of motion for the system

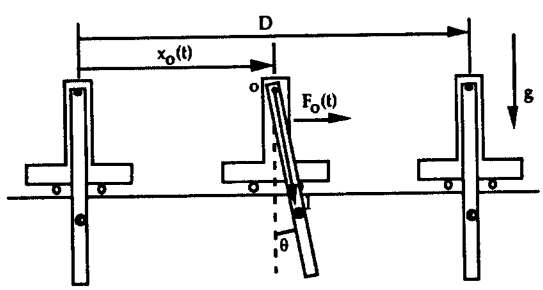

Figure 7 The System : Movement of a Compound Pendulum

(assuming small angular motions θ (t)):

$$\ddot{x}_o + \left(\frac{J_o}{ml}\right)\ddot{\theta} + g\theta = 0 \quad (34)$$

$$(m_o + m)\ddot{x}_o + ml\ddot{\theta} = F_o(t) \quad (35)$$

Rewriting (34), we get:

$$\ddot{\theta} + \omega_n^2\theta = -\left(\frac{ml}{J_o}\right)\ddot{x}_o \quad (36)$$

where $\omega_n = \sqrt{\frac{mgl}{J_o}}$

As before, we use x(t) from (24). Thus (36) becomes:

$$\ddot{\theta} + \omega_n^2\theta = -\left(\frac{D\omega_n^2 ml}{2\alpha^2 J_o}\right)\cos\left(\frac{\omega_n}{\alpha}t\right) \quad (37)$$

Comparing this to (24) shows that we can replace $\left(\frac{D}{2l}\right)$ in (24) and (27) by $\left(\frac{Dml}{2J_o}\right)$ to get the results for the present case.

$$\theta\,(t) = \frac{Dml}{2J_o}\left(\frac{1}{\alpha^2 - 1}\right)\left[\cos(\omega_n t) - \cos\left(\frac{\omega_n}{\alpha}t\right)\right] \qquad 0 \le t \le t_f \quad (38)$$

$$= 0 \qquad t > t_f$$

Similarly, we can determine θ_{max} (here for α = 3):

$$\theta_{max} = 0.096\frac{Dml}{J_o}, \quad \text{rad.s} \quad (\alpha = 3) \quad (39)$$

and again, the final time is:

$$t_f = \left(\frac{\alpha\pi}{\omega_n}\right), \quad \alpha = 3, 5, 7, ..., (2n+1) \quad (40)$$

To determine the force required to give the motion (38), we note the equation (35). Inserting x(t) from (3) and θ (t) from (38) to (35), we get :

$$\frac{2F_o}{Dm\omega_n^2} = -\frac{1}{\beta\,(\alpha^2 - 1)}\cos(\omega_n t)$$

$$+\frac{1}{\alpha^2}\left[(R_m + 1) + \frac{1}{\beta\,(\alpha^2 - 1)}\right]\cos\left(\frac{\omega_n}{\alpha}t\right) \quad (41)$$

where $R_m = \frac{m_o}{m}$ and $\beta = \frac{J_o}{ml^2}$

Notice that (41) is a function of both the mass ratio R_m and the moment of inertia ratio β. In the case of the simple pendulum, $J_o = ml^2$. Thus β = 1. If we take R_m = 0.2 and α = 3, the plot of (41) is that of Figure 8.

The equation for the force required to move the compound pendulum (41) is a function of the mass ratio R_m and the inertia ratio β.

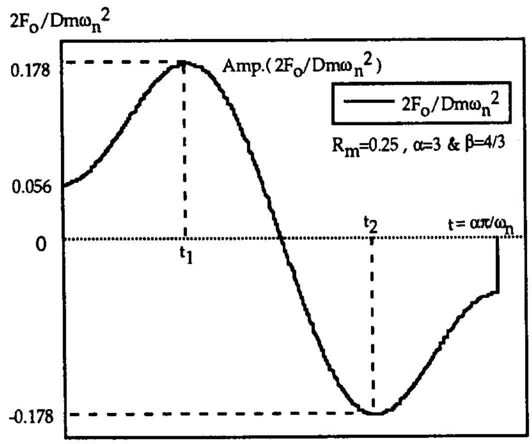

Figure 8 The Force Required to Initiate the Motion $x_o(t)$ for the Movement of a Pendulum Movement

The minimum value of β is 1.0 (for a simple pendulum of length l). Thus $R_m \geq 0$ and $\beta \geq 1$. From (41), the initial and final values of $(\frac{2F_o}{Dm\omega_n^2})$ are:

Initial and Final

$$\left(\frac{2F_o}{Dm\omega_n^2}\right)=\frac{1}{\alpha^2}\left[-\frac{1}{\beta}+1+R_m\right] \qquad (42)$$

Clearly if $\beta \geq 1$ and $R_m \geq 0$, this quantity is greater than or equal to zero.

Conclusions

Simply stated, a mass can be moved from point A to point B without inducing vibration if we use a [1 - cos(ωt)] specified motion where ω is one third (or one fifth ...) of the natural frequency ω_n of the system which is being moved. Clearly, the motion shape [1 - cos(ωt)] term must be applied during the time interval from t = 0 to $t = \frac{\alpha\pi}{\omega_n}$. Taking $\alpha = 3$ gives the smoothest transition from A to B in the minimum time. In the case of the pendulum, the cost of using $\alpha = 3$ is that the amplitude of the pendulum is highest at that value of α .

A number of questions remain unanswered at this point. For example we have considered only undamped single degree of freedom systems here. Future research will determine whether the ideas here can be expanded to include the systems with damping or the systems with several degrees of freedom [5]. If there is damping in a single degree of freedom system, we will not be able to bring $\dot{x}$ to zero at the end of the cycle with the open-loop procedure outlined here. However, the procedure given here could be used in conjunction with a mechanical capture system or a closed loop control to achieve the desired goal.

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