Definition 2.3

The second difference of the function f is denoted by $\Delta^2 f(n)$ and is defined as the difference of the first difference, that is,

$$\Delta^2 f(n) = \Delta f(n+1) - \Delta f(n).$$

Remark. f is discrete convex if $\Delta^2 f(n) \geq 0$ (For additional definitions of discrete and continuous kinds of convexity see Ponstein [15] and Kumin [10], [11]).

Theorem 2.1

The product of two positive (strictly) decreasing and discrete convex functions is also a decreasing and discrete convex function.

Proof. Assume that $F(c)$ and $G(c)$ are positive (strictly) decreasing, discrete convex functions. Let $H(c) = F(c)G(c)$.

$$\Delta H(c) = F(c+1)G(c+1) - F(c)G(c)$$

$$= F(c+1)\Delta G(c) + G(c)\Delta F(c) < 0.$$

Hence, $H(c)$ is strictly decreasing.

Secondly, we show $\Delta^2 H(c) > 0$.

$$\Delta^2 H(c) = F(c+2)\Delta G(c+1) - F(c+1)\Delta G(c)$$

$$+G(c+1)\Delta F(c+1) - G(c)\Delta F(c)$$

$$= F(c+2)\Delta^2 G(c) + \Delta G(c)\Delta F(c+1)$$

$$+G(c+1)\Delta^2 F(c) + \Delta F(c)\Delta G(c) > 0,$$

since $F(c)$ and $G(c)$ are discrete convex functions in c.

Therefore the product of two (strictly) decreasing and discrete convex functions is also a decreasing and discrete convex function.

3. A Discrete Convexity Result for the Erlang delay formula.

Theorem 3.1

The Erlang delay formula (or Erlang "B" formula) is a decreasing and discrete convex function in $c > 1$.

Proof. See Choi [1] and Jagers and Van Doorn [9].

4. The Discrete Convexity of Some Performance Measures of an M/M/c Queueing System

Theorem 4.1

In the M/M/c queue, the average number of customers in the queue is a decreasing and discrete convex function of c.

Proof. The average number of customers in the queue is:

$$L_q = B\frac{\frac{\rho}{c}}{1-\frac{\rho}{c}}$$

$$= B\left(\frac{\rho}{c-\rho}\right)$$

where B is the Erlang delay formula.

We have already shown by Theorem 2.1 that B is decreasing and discrete convex in c.

Let

$$F(c) = \frac{\rho}{c-\rho}$$

and G(c) be the Erlang delay formula. Since $\Delta F(c) < 0$, $F(c)$ is a decreasing function.

$$\Delta^2 F(c) = \frac{\rho}{c+2-\rho} - \frac{\rho}{c+1-\rho} - \frac{\rho}{c+1-\rho} + \frac{\rho}{c-\rho}$$

$$= \frac{\rho[-(c+2-\rho)(c-\rho)-(c-\rho)+(c+2-\rho)(c+1-\rho)]}{(c+2-\rho)(c+1-\rho)(c-\rho)}$$

$$= 2\frac{\rho}{(c+2-\rho)(c+1-\rho)(c-\rho)}$$

$$> 0.$$

Hence $F(c)$ is decreasing and discrete convex.

Since L_q is the product of two decreasing and discrete convex functions in c, L_q is also decreasing and discrete convex in c.

Theorem 4.2

In the M/M/c queue, the average waiting time of a customer is a decreasing and discrete convex function of c.

Proof. The average waiting time of a customer can be expressed using Little's formula by:

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{1}{\mu(c-\rho)} B$$

Let

$$F(c) = \frac{1}{c-\rho} \quad \text{and} \quad G(c) = B$$

Since B is the Erlang delay formula, $G(c)$ is a decreasing and discrete convex function. It is clear that $\Delta F(c) < 0$, thus $F(c)$ is decreasing.

$$\Delta^2 F(c) = \frac{1}{c+2-\rho} - \frac{1}{c+1-\rho} - \frac{1}{c+1-\rho} + \frac{1}{c-\rho}$$

$$= \frac{-(c-\rho)-(c+2-\rho)(c-\rho)+(c+2-\rho)(c+1-\rho)}{(c+2-\rho)(c+1-\rho)(c-\rho)}$$

$$= \frac{2}{(c+2-\rho)(c+1-\rho)(c-\rho)}$$

$$> 0.$$

Thus, $F(c)$ is a discrete convex function in c.

Since μ is positive and W_q is the product of two decreasing and discrete convex functions, W_q is also decreasing and discrete convex in c .

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